# Probability and Statistics

## Exercise sheet 1 - Solutions

**MC 1.1.** Let $A, B \subseteq \Omega$. Which of the following does **not** hold? (Exactly one answer is correct.)

(a) $(A \setminus B)^c = B \cup A^c$.

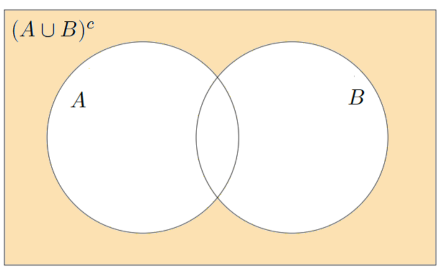
(b) $(A \cup B)^c = A^c \cap B^c$.

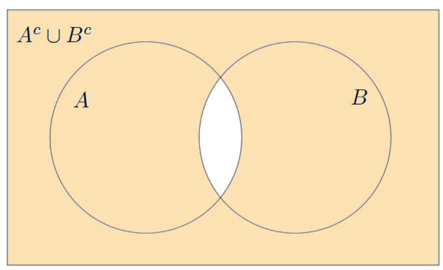
(c) $A \setminus B^c = A \cap B$.

(d) $(A \cup B)^c = A^c \cup B^c$.

**Solution:** (d) doesn't hold. Let $\omega \in B \setminus A = B \cap A^c$. Then we have $\omega \in A^c$ and so $\omega \in A^c \cup B^c$. We also have $\omega \in B$ and so it holds $\omega \in A \cup B$, which gives $\omega \notin (A \cup B)^c$.

Graphically, we have

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**MC 1.2.** Let $\Omega := \{\omega_1, \omega_2, \omega_3\}$. Which of the following does **not** define a $\sigma$-algebra on $\Omega$? (Exactly one answer is correct.)

(a) $\mathcal{F}_1 := \big\{\emptyset, \Omega\big\}$.

(b) $\mathcal{F}_2 := \big\{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\big\}$.

(c) $\mathcal{F}_3 := \big\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\big\}$.

(d) $\mathcal{F}_4 := \big\{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \Omega\big\}$.

**Solution:** $\mathcal{F}_3$ is not a $\sigma$-algebra. For instance, we have $\{\omega^1\} \in \mathcal{F}_3$, but $\{\omega_1\}^c = \{\omega_2, \omega_3\} \notin \mathcal{F}_3$.

**MC 1.3.** Let $\Omega := \{0, 1\}$, and $\mathcal{F} := 2^\Omega$. Which of the following define a probability measure on $\Omega$? (The number of correct answers is between 0 and 4.)

(a) $\mathbb{P}[\emptyset] = \mathbb{P}[\{0\}] = \mathbb{P}[\{1\}] = \mathbb{P}[\{0, 1\}] = \frac{1}{4}$.

(b) $\mathbb{P}[\emptyset] = \mathbb{P}[\{0\}] = \mathbb{P}[\{1\}] = 0$ and $\mathbb{P}[\{0,1\}] = 1$.

(c) $\mathbb{P}[\emptyset] = 0, \mathbb{P}[\{0\}] = \mathbb{P}[\{1\}] = \frac{1}{2}$, and $\mathbb{P}[\{0,1\}] = 1$.

(d) $\mathbb{P}[\emptyset] = 0, \mathbb{P}[\{0\}] = \frac{1}{4}, \mathbb{P}[\{1\}] = \frac{1}{2}$, and $\mathbb{P}[\{0,1\}] = \frac{3}{4}$.

**Solution:**
(a) is not a probability measure. For instance, every probability measure must satisfy $\mathbb{P}[\emptyset] = 0$.
(b) is not a probability measure. For instance, every probability measure must satisfy $\mathbb{P}[\{0,1\}] = \mathbb{P}[\{0\} \cup \{1\}] = \mathbb{P}[\{0\}] + \mathbb{P}[\{1\}]$.
(c) is a probability measure.
(d) is not a probability measure. For example, every probability measure must satisfy $\mathbb{P}[\Omega] = \mathbb{P}[\{0,1\}] = 1$.

**Exercise 1.4. [Settlers of Catan]** We are playing the board game Settlers of Catan. The game board consists of landscapes that are labeled with integers between 2 and 6 or between 8 and 12. In each round, two dice are rolled and those landscapes whose number matches the sum of the dice rolls yield resources.

(a) Define the sample space $\Omega := \{(w_1, w_2) \,|\, w_1, w_2 \in \{1,2,3,4,5,6\}\}$. Identify the event {the landscapes with number 9 yield resources} as a subset of $\Omega$.

(b) Which landscapes (i.e., which numbers) are expected to yield resources most frequently and least frequently? Why?

(c) A player has a choice: Either they receive future resources from a landscape with number 8 or from both landscapes with number 4 and 12. What should they choose and why? (We assume that the type of resource does not influence the decision.)

**Solution:**

(a) The desired subset is
$$\Omega_9 = \big\{(3,6), (4,5), (5,4), (6,3)\big\}.$$

(b) We proceed as follows: The probability space consists of 36 elements. These can be grouped by the sum of the dice rolls into 11 sets ($\Omega_2$, $\Omega_3$, …, $\Omega_{12}$), where the indices correspond to the sum of the dice. By counting the elements in each $\Omega_i$, we find
$$|\Omega_2| = 1, |\Omega_3| = 2, \ldots, |\Omega_7| = 6, |\Omega_8| = 5, \ldots, |\Omega_{12}| = 1.$$
The more elements a set has, the more frequently the corresponding number will appear as a sum of dice rolls. Using the Laplace model, we can explicitly compute the probabilities:
$$\mathbb{P}[\Omega_2] = 1/36, \mathbb{P}[\Omega_3] = 2/36, \ldots, \mathbb{P}[\Omega_7] = 6/36, \mathbb{P}[\Omega_8] = 5/36, \ldots, \mathbb{P}[\Omega_{12}] = 1/36.$$
The answer is thus: The most productive landscapes are the ones with number 6 and 8, while the least productive ones are the ones with number 2 and 12. Note that there are no landscapes with number 7.

(c) Using part (b), we know that $|\Omega_8| = 5$, $|\Omega_4| = 3$, and $|\Omega_{12}| = 1$. This allows us to compute the following probabilities: $\mathbb{P}[\Omega_8] = 5/36$, and since $\Omega_4$ and $\Omega_{12}$ are disjoint, we obtain $\mathbb{P}[\Omega_4 \cup \Omega_{12}] = \mathbb{P}[\Omega_4] + \mathbb{P}[\Omega_{12}] = 4/36$. Since the probability of obtaining resources is higher in the first option, the player should choose this option.

**Exercise 1.5. [Biased coins]** We assume that we have two biased coins, gold and silver, in an urn. The probability that the gold coin lands on heads is $p_g \in (0,1)$, and for silver, it is $p_s \in (0,1)$. In each trial, a coin is drawn from the urn, tossed, and then returned to the urn. We conduct the random experiment twice.

(a) Specify an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (We assume that the gold and silver coin are each drawn with probability 1/2.)

(b) Which element of $\mathcal{F}$ corresponds to the event $A = \{\text{The first coin drawn is silver}\}$.

(c) Which element of $\mathcal{F}$ corresponds to the event $B = \{\text{Heads is obtained twice}\}$.

(d) Compute $\mathbb{P}[A]$, $\mathbb{P}[B]$, and $\mathbb{P}[A \cap B]$.

**Solution:**

(a) First, we define a sample space that includes all possible outcomes. Each time the experiment is conducted, the drawn coin can be gold ($g$) or silver ($s$), and the toss can result in heads ($H$) or tails ($T$). Thus, the outcome of a single trial can be represented as an element of $\{(g,H),(g,T),(s,H),(s,T)\}$. Since the experiment is conducted twice, the sample space is:

$$\Omega = \{(g,H),(g,T),(s,H),(s,T)\}^2.$$

The elements of $\Omega$ take the form $\omega = ((m_1,x_1),(m_2,x_2))$, where $m_1,m_2 \in \{g,s\}$ and $x_1,x_2 \in \{H,T\}$. For instance, $((g,H),(s,H)) \in \Omega$ represents the outcome where the first coin is gold and lands on heads, and the second coin is silver and lands on heads.

We choose the power set of $\Omega$ as the $\sigma$-algebra, so $\mathcal{F} = 2^\Omega$. In particular, for all $\omega \in \Omega$, the event $\{\omega \text{ occurs}\}$ is in $\mathcal{F}$, meaning $\{\omega\} \in \mathcal{F}$.

Next, we define an appropriate probability measure $\mathbb{P}$ in two steps. First, for each outcome $\omega = \big((m_1,x_1),(m_2,x_2)\big) \in \Omega$, we define:

$$\mathbb{P}\big[\{((m_1,x_1),(m_2,x_2))\}\big] = \begin{cases} \frac{1}{4}p_{m_1}p_{m_2} & \text{if } x_1 = x_2 = H, \\ \frac{1}{4}(1-p_{m_1})p_{m_2} & \text{if } x_1 = T, x_2 = H, \\ \frac{1}{4}p_{m_1}(1-p_{m_2}) & \text{if } x_1 = H, x_2 = T, \\ \frac{1}{4}(1-p_{m_1})(1-p_{m_2}) & \text{if } x_1 = x_2 = T. \end{cases}$$

For example, the event {The first coin is gold and lands on heads, and the second coin is silver and lands on heads} has probability $\frac{1}{4} \times p_g \times p_s$. The factor $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$ arises because the first coin is gold with probability $\frac{1}{2}$ and the second coin is silver with probability $\frac{1}{2}$. In the second step, for any event $A \in \mathcal{F}$, we define:

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\{\omega\}].$$

Since $\Omega$ is finite, it is easy to verify that the mapping $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ is indeed a probability measure on $(\Omega, \mathcal{F})$, particularly satisfying:

$$\mathbb{P}[\Omega] = \sum_{\omega \in \Omega} \mathbb{P}[\{\omega\}] = 1.$$

(b) The event {The first drawn coin is silver} corresponds to:

$$\begin{aligned} A &= \big\{((m_1, x_1), (m_2, x_2)) \in \Omega : m_1 = s\big\} \\ &= \big\{((s,H),(g,H)), ((s,H),(g,T)), ((s,H),(s,H)), ((s,H),(s,T)), \\ &\qquad ((s,T),(g,H)), ((s,T),(g,T)), ((s,T),(s,H)), ((s,T),(s,T))\big\} \in \mathcal{F}. \end{aligned}$$

(c) The event {Heads is obtained twice} corresponds to:

$$\begin{aligned} B &= \big\{((m_1, x_1), (m_2, x_2)) \in \Omega : x_1 = x_2 = H\big\} \\ &= \big\{((g,H),(g,H)), ((g,H),(s,H)), ((s,H),(g,H)), ((s,H),(s,H))\big\} \in \mathcal{F}. \end{aligned}$$

(d) We compute:

$$\begin{aligned} \mathbb{P}[A] &= \sum_{\omega \in A} \mathbb{P}[\{\omega\}] = \sum_{m_2 \in \{g,s\}; x_1, x_2 \in \{H,T\}} \mathbb{P}\big[\{((s,x_1),(m_2,x_2))\}\big] \\ &= \sum_{x_1, x_2 \in \{H,T\}} \mathbb{P}\big[\{((s,x_1),(g,x_2))\}\big] + \sum_{x_1, x_2 \in \{H,T\}} \mathbb{P}\big[\{((s,x_1),(s,x_2))\}\big] \\ &= \frac{1}{4} \underbrace{\big(p_s \times p_g + (1-p_s) \times p_g + p_s \times (1-p_g) + (1-p_s) \times (1-p_g)\big)}_{=1} \\ &\quad + \frac{1}{4} \underbrace{\big(p_s \times p_s + (1-p_s) \times p_s + p_s \times (1-p_s) + (1-p_s) \times (1-p_s)\big)}_{=1} \\ &= \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}[B] &= \sum_{\omega \in B} \mathbb{P}[\{\omega\}] = \sum_{m_1, m_2 \in \{g,s\}} \underbrace{\mathbb{P}\big[\{((m_1,H),(m_2,H))\}\big]}_{=\frac{1}{4} \times p_{m_1} \times p_{m_2}} \\ &= \frac{1}{4} \times \big((p_g)^2 + 2 \times p_s \times p_g + (p_s)^2\big) = \frac{(p_g + p_s)^2}{4}. \end{aligned}$$

Furthermore, since $A \cap B = \big\{((s,H),(g,H)\big), \big((s,H),(s,H))\big\} \in \mathcal{F}$, we get:

$$\begin{aligned} \mathbb{P}[A \cap B] &= \sum_{\omega \in A \cap B} \mathbb{P}[\{\omega\}] = \mathbb{P}\big[\{((s,H),(g,H))\}\big] + \mathbb{P}\big[\{((s,H),(s,H))\}\big] \\ &= \frac{1}{4} \times p_s \times p_g + \frac{1}{4} \times p_s \times p_s = \frac{p_s \times (p_g + p_s)}{4}. \end{aligned}$$

**Exercise 1.6. [Properties of a $\sigma$-algebra]**

(a) [De Morgan's Law] Let $(A_i)_{i\geq 1}$ be a sequence of arbitrary sets. Show that the following holds:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} (A_i)^c.$$

Let $\mathcal{F}$ be a $\sigma$-algebra on $\Omega$.

(b) Show that $\emptyset \in \mathcal{F}$.

(c) Let $(A_i)_{i\geq 1}$ be a sequence of events, i.e., $A_i \in \mathcal{F}$ for all $i \geq 1$. Show that

$$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}.$$

(d) Let $A, B \in \mathcal{F}$. Show that $A \cup B \in \mathcal{F}$.

(e) Let $A, B \in \mathcal{F}$. Show that $A \cap B \in \mathcal{F}$.

**Solution:**

(a) We prove De Morgan's law by showing both inclusions.
$\subseteq$: Let $\omega \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c$. For all $j \in \mathbb{N}$, we clearly have $A_j \subseteq \bigcup_{i=1}^{\infty} A_i$, which implies that for all $j \in \mathbb{N}$,

$$\omega \in (A_j)^c.$$

This implies that $\omega \in \bigcap_{j=1}^{\infty} (A_i)^c$. Thus, $\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{j=1}^{\infty} (A_i)^c$.
$\supseteq$: Let $\omega \in \bigcap_{i=1}^{\infty} (A_i)^c$. This means that for all $1 \leq j < \infty$,

$$\omega \in (A_j)^c,$$

or equivalently, $\omega \notin A_j$. This implies $\omega \notin \bigcup_{j=1}^{\infty} A_j$ and thus $\omega \in \left(\bigcup_{j=1}^{\infty} A_j\right)^c$. In other words, $\bigcap_{j=1}^{\infty} (A_i)^c \subseteq \left(\bigcup_{i=1}^{\infty} A_i\right)^c$.

(b) Since $\Omega \in \mathcal{F}$, it follows that

$$\emptyset = \Omega^c \in \mathcal{F}.$$

(c) Let $A_1, A_2, \ldots \in \mathcal{F}$. Then, $A_1^c, A_2^c, \ldots \in \mathcal{F}$. Thus, it follows that $\bigcup_{i=1}^{\infty} (A_i)^c \in \mathcal{F}$. By De Morgan's law, we obtain:

$$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} (A_i)^c\right)^c \in \mathcal{F}.$$

(d) Let $A, B \in \mathcal{F}$. We define $A_1 \coloneqq A$, $A_2 \coloneqq B$, and for all $i \geq 3$, $A_i \coloneqq \emptyset \in \mathcal{F}$. Then, we obtain:

$$A \cup B = \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

(e) Let $A, B \in \mathcal{F}$, so that $A^c, B^c \in \mathcal{F}$. From (d), we now obtain $A^c \cup B^c \in \mathcal{F}$. It follows that:

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{F},$$

where we have applied De Morgan's law for two sets.

**Exercise 1.7. [Properties of a probability measure]** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Show that $\mathbb{P}[\emptyset] = 0$.

(b) Let $k \geq 1$, and let $A_1, \ldots, A_k$ be $k$ pairwise disjoint events. Show that

$$\mathbb{P}[A_1 \cup \cdots \cup A_k] = \mathbb{P}[A_1] + \cdots + \mathbb{P}[A_k].$$

(c) Let $A$ be an event. Show that $\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$.

(d) Let $A$ and $B$ be two arbitrary events (not necessarily disjoint). Show that the addition rule

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

holds.

**Solution:**

(a) Define $x = \mathbb{P}[\emptyset]$. We already know that $x \in [0, 1]$, since $x$ is the probability of an event. Now, we define $A_1 = A_2 = \cdots = \emptyset$ and thus obtain

$$\emptyset = \bigcup_{i=1}^{\infty} A_i.$$

Since the events $A_i$ are disjoint, countable additivity implies that

$$\sum_{i=1}^{\infty} P[A_i] = \mathbb{P}[\emptyset].$$

Since $\mathbb{P}[A_i] = x$ for every $i$ and $P[\emptyset] \leq 1$, we obtain

$$\sum_{i=1}^{\infty} x \leq 1,$$

and therefore $x = 0$.

(b) Define $A_{k+1} = A_{k+2} = \cdots = \emptyset$. In this way, we have

$$A_1 \cup \cdots \cup A_k = A_1 \cup \cdots \cup A_k \cup \emptyset \cup \emptyset \cup \cdots = \bigcup_{i=1}^{\infty} A_i.$$

Since the events $A_i$ are pairwise disjoint, we apply countable additivity as follows:

$$\begin{aligned} \mathbb{P}[A_1 \cup \cdots \cup A_k] &= \mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \\ &= \sum_{i=1}^{\infty} \mathbb{P}[A_i] \\ &= \mathbb{P}[A_1] + \cdots + \mathbb{P}[A_k] + \underbrace{\sum_{i>k} \underbrace{\mathbb{P}[A_i]}_{=0}}_{=0}. \end{aligned}$$

(c) By the definition of the complement, we have $\Omega = A \cup A^c$, and thus

$$1 = \mathbb{P}[\Omega] = \mathbb{P}[A \cup A^c].$$

Since the two events $A$, $A^c$ are disjoint, part (b) implies that

$$1 = \mathbb{P}[A] + \mathbb{P}[A^c].$$

(d) $A \cup B$ is the disjoint union of $A$ and $B \setminus A$. Using part (b), we get

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B \setminus A]. \tag{1}$$

Since $B = (B \cap A) \cup (B \cap A^c) = (B \cap A) \cup (B \setminus A)$, where $B \cap A$ and $B \setminus A$ are disjoint, we obtain

$$\mathbb{P}[B] = \mathbb{P}[B \cap A] + \mathbb{P}[B \setminus A],$$

which implies that $\mathbb{P}[B \setminus A] = \mathbb{P}[B] - \mathbb{P}[A \cap B]$. Substituting this into equation (1), we obtain the result.