Probability and Statistics

Exercise sheet 2

MC 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Which of the following statements is true? (Exactly one answer is correct.)

(a) $\forall A, B \in \mathcal{F} : \mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$.

(b) $\forall A, B \in \mathcal{F} : \mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B]$.

(c) $\forall A, B \in \mathcal{F} : \mathbb{P}[A \cap B] \leq \mathbb{P}[A]\mathbb{P}[B]$.

(d) None of these (in)equalities always hold.

MC 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A, B \in \mathcal{F}$ with $\mathbb{P}[A] = 0.5$ and $\mathbb{P}[B] = 0.8$. Which of the following statements is always true? (The number of correct answers is between 0 and 4.)

(a) $\mathbb{P}[A \cap B] = 0.5$.

(b) $\mathbb{P}[A \cap B] \leq 0.5$.

(c) $\mathbb{P}[A \cap B] \geq 0.3$.

(d) $\mathbb{P}[A \cap B] \geq 0.5$.

MC 2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A, B \in \mathcal{F}$. In which cases does the following hold?

$$\mathbb{P}[A \setminus B] = \mathbb{P}[A] - \mathbb{P}[B].$$

(The number of correct answers is between 0 and 4.)

(a) Always.

(b) If $B \subseteq A$.

(c) If $\mathbb{P}[A] \geq \mathbb{P}[B]$.

(d) If $A = \Omega$.

Exercise 2.4. Let A and B be two events with

$$\mathbb{P}[A^c] = \frac{1}{2}, \quad \mathbb{P}[B^c] = \frac{1}{2}, \quad \mathbb{P}[A^c \cap B^c] = p.$$

(a) Find as a function of p the probabilities $\mathbb{P}[A \cap B]$, $\mathbb{P}[A \cap B^c]$ and $\mathbb{P}[A^c \cap B]$. What are the possible values of p for the above to be well-defined?

(b) Find as a function of p the probability that at most i of the two events A and B occur, where $i \in \{0, 1, 2\}$.

Exercise 2.5. [Radio signals] Four signals are transmitted sequentially over a communication channel. Each signal is transmitted either correctly or incorrectly. We define the sample space Ω as the set of all 0-1 sequences of length 4 as follows:

$$\Omega = \{\omega = (x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \in \{0, 1\}\},$$

i.e.,

$$\Omega = \{(0,0,0,0), (1,0,0,0), (0,1,0,0), \ldots, (1,1,1,1)\},$$

and we interpret $x_i = 1$ as "the i-th signal is transmitted correctly" and $x_i = 0$ as "the i-th signal is transmitted incorrectly" for $i = 1, \ldots, 4$. We consider the following events:

- $A := \{\text{Exactly one signal is transmitted incorrectly}\}$,
- $B := \{\text{At least two signals are transmitted correctly}\}$,
- $C := \{\text{At most two signals are transmitted correctly}\}$.

(a) Write the events A, B, and C as subsets of Ω.

(b) Describe in words the events $B \cap C$, $A \cup B$, and $A^c \cap C^c$.

(c) Calculate the probabilities of the events A, B, and C assuming that all elementary events $(x_1, x_2, x_3, x_4) \in \Omega$ are equally likely. What model are we using here?

Exercise 2.6. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space and let $(B_i)_{i=1}^{\infty}$ be a sequence of almost surely occurring events, i.e., $\mathbb{P}[B_i] = 1$ for all $i \geq 1$. Show that

$$\mathbb{P}\Big[\bigcap_{i=1}^{\infty} B_i\Big] = 1,$$

i.e., almost surely, all (infinitely many) events occur.

Exercise 2.7. [Birthdays] We have a class of $n \in \mathbb{N}$ students whose birthdays are randomly distributed throughout the year. To simplify, assume that a year has 365 days and that a birthday is equally likely to fall on any of the 365 days (this is not true in reality, see e.g. statistics in the UK or in Switzerland). Further, assume that the birthdays of the students are independent of each other, i.e., there are for instance no twins. (For now, you can understand "independence" intuitively or as you learned it in high school. More precisely, the assumption is that all distributions of the n students' birthdays over the 365 days are equally likely.) Finally, assume that $1 < n < 365$.

(a) Calculate the probability that there is (at least) one student whose birthday is today.

(b) Alice and Bob are two students from this class. What is the probability that they both have their birthdays today?

(c) What is the probability that Alice and Bob have the same birthday?

(d) What is the probability that (at least) two students have the same birthday?

(e) What is the probability that (at least) two students have their birthdays today?

Exercise 2.8. [Simpson's paradox] We are interested in the probability of a student's success in an entrance exam for two departments of a university. Consider the following events:

$$A := \{\text{The student is male}\},$$
$$B := \{\text{The student applied to Department I}\},$$
$$C := \{\text{The student was accepted}\}.$$

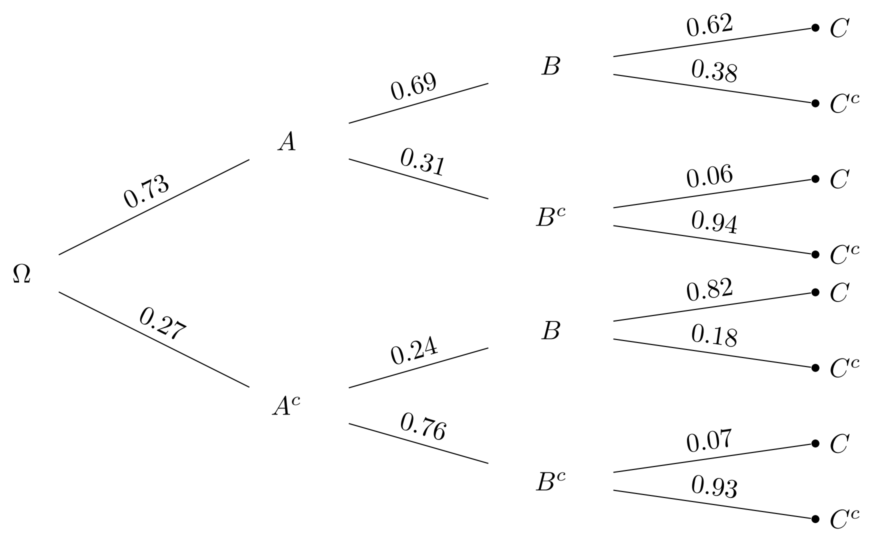
Therefore,

$$A^c = \{\text{The student is not male}\},$$
$$B^c = \{\text{The student applied to Department II}\},$$
$$C^c = \{\text{The student was not accepted}\}.$$

We know the following probabilities:

$$\mathbb{P}[A] = 0.73,$$
$$\mathbb{P}[B|A] = 0.69, \quad \mathbb{P}[B|A^c] = 0.24,$$
$$\mathbb{P}[C|A \cap B] = 0.62, \quad \mathbb{P}[C|A^c \cap B] = 0.82,$$
$$\mathbb{P}[C|A \cap B^c] = 0.06, \quad \mathbb{P}[C|A^c \cap B^c] = 0.07.$$

Graphically, this is represented as follows:



(a) Explain in words what the above conditional probabilities represent. Do you think that individuals who are not male are disadvantaged in the selection process? Why or why not?

(b) Calculate $\mathbb{P}[C|A]$ and $\mathbb{P}[C|A^c]$, i.e., the acceptance probabilities for males and for individuals who are not male. Does this match your answer in (a)? Can you explain what is happening here?

Exercise 2.9. [Nadal vs. Federer] We analyze a tennis match between Roger Federer and Rafael Nadal. The match is played under the "best of 3" rule, i.e., the winner is the first to win two sets (a maximum of 3 sets are played). We assume that Federer wins each set – independently of the others – with probability $p = \frac{1}{3}$. Let A denote the event that Federer wins the first set and B denote the event that Federer wins the match (i.e., wins two sets).

(a) Express $A \cup B$, $A^c \cap B$, $A \cap B^c$ and $A \setminus B$ in words. Calculate the conditional probabilities $P[B^c|A]$, $P[B|A]$ and $P[B|A^c]$.

(b) Calculate the probability that Federer wins the match.

(c) Calculate the conditional probabilities $P[A|B]$ and $P[A|B^c]$.