Probability and Statistics

Exercise sheet 3

MC 3.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let A, B, and C be events with $\mathbb{P}[A \cap B] > 0$ and $\mathbb{P}[C] > 0$. We assume that $\mathbb{P}[A|B] > \mathbb{P}[A]$ and $\mathbb{P}[A|C] > \mathbb{P}[A]$. Which of the following holds? (Exactly one answer is correct.)

(a) $\mathbb{P}[A|B \cap C] > \mathbb{P}[A]$.

(b) $\mathbb{P}[B] = \mathbb{P}[C]$.

(c) $\mathbb{P}[B|A] > \mathbb{P}[B]$.

(d) None of the above.

MC 3.2. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{F} = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3\}\}$. Which of the following define random variables on $(\Omega, \mathcal{F})$? (The number of correct answers is between 0 and 4.)

(a) $X_1(\omega_1) = 1$, $X_1(\omega_2) = 2$, $X_1(\omega_3) = 3$.

(b) $X_2(\omega_1) = 1$, $X_2(\omega_2) = 1$, $X_2(\omega_3) = 2$.

(c) $X_3(\omega_1) = 1$, $X_3(\omega_2) = 2$, $X_3(\omega_3) = 2$.

(d) $X_4(\omega_1) = 1$, $X_4(\omega_2) = 1$, $X_4(\omega_3) = 1$.

MC 3.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let A, B, and C be events in $\mathcal{F}$. Which of the following statements are always true? (The number of correct answers is between 0 and 4.)

(a) If A and B as well as A and C are independent, then A and $B \cap C$ are also independent.

(b) If A and B as well as B and C are independent, then A and C are also independent.

(c) If A, B, and C are independent, then A and $B \cap C$ are also independent.

(d) If A and A are independent, then $\mathbb{P}[A] = 1$ or $\mathbb{P}[A] = 0$.

MC 3.4. Let X and Y be two random variables taking values in $\{1, \dots, 6\}$ and representing two independent rolls of a die. Which of the following pairs of events are independent? (The number of correct answers is between 0 and 4.)

(a) $\{X \text{ is odd}\}$, $\{X + Y \text{ is even}\}$.

(b) $\big\{X \in \{1, 3\}\big\}$, $\{X + Y = 5\}$.

(c) $\{X = 1\}$, $\{X + Y = 4\}$.

(d) $\{X = 1\}$, $\{X + Y = 13\}$.

Exercise 3.5. Let X be a random variable with the distribution function

$$F(a) = \begin{cases} 0, & a < 0, \\ \frac{a}{2}, & 0 \leq a < 1, \\ \frac{2}{3}, & 1 \leq a < 2, \\ \frac{a+1}{4}, & 2 \leq a < 3, \\ 1, & 3 \leq a. \end{cases}$$

(a) Plot this distribution function.

(b) Determine the following probabilities: $\mathbb{P}[X < 1]$, $\mathbb{P}[X = 2]$, $\mathbb{P}[X = 3]$, $\mathbb{P}[1 < X \leq 2]$, $\mathbb{P}[1 \leq X < 2]$ and $\mathbb{P}[X \geq 3/2]$.

Exercise 3.6. [Riemann zeta function] Let X be a discrete random variable with values in $\mathbb{N} = \{1, 2, 3, \ldots\}$. The distribution of X is given by

$$\mathbb{P}[X = n] = \frac{n^{-s}}{\zeta(s)}, \quad n \in \mathbb{N},$$

where $s > 1$ is a parameter of the distribution, and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ denotes the Riemann zeta function. For a number $m \in \mathbb{N}$, we define the event E_m as $\{X$ is divisible by m without remainder$\}$, or equivalently, $\{$There exists a $k \in \mathbb{N}$ such that $X = km\}$.

(a) Show that $\mathbb{P}[E_m] = m^{-s}$ for all $m \in \mathbb{N}$.

(b) Let p and q be two distinct prime numbers. Show that E_p and E_q are independent.

Hint: A number n is divisible by two different prime numbers p and q if and only if n is divisible by pq.

(c) Determine $\mathbb{P}\Big[\bigcap_{p \text{ prime}} E_p^c\Big]$.

Exercise 3.7. [First six] Two players, Anja and Beatrice, take turns rolling a (fair) die until a six appears. Anja starts rolling. The player who rolls the first six wins the game. Determine the probabilities of winning for both players.

Exercise 3.8. [Construction of random variables] The goal of this problem is to construct random variables from a sequence of independent coin flips. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(X_i)_{i \geq 1}$ be an infinite sequence of independent, Bernoulli(1/2)-distributed random variables. We consider the following algorithm:

$$\begin{aligned} & i \coloneqq 1 \\ & \texttt{while} \quad (X_i = X_{i+1} = 1): \\ & \qquad i \coloneqq i + 2 \\ & Z \coloneqq X_i + 2 \times X_{i+1} \\ & \texttt{return} \quad Z \end{aligned}$$

(a) Show that the algorithm always terminates after a finite number of steps with probability 1.

(b) Show that Z is a uniformly distributed random variable in $\{0, 1, 2\}$.

(c) **[Bonus]** Provide an algorithm that outputs a Bernoulli(1/5)-distributed random variable.