Probability and Statistics

Exercise sheet 4 - Solutions

MC 4.1. [Even-indexed Poisson(1)] Let X be a random variable taking values in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ with

$$\mathbb{P}[X = k] = \begin{cases} c/k!, & k \in \mathbb{N}_0,\ k \text{ even}, \\ 0, & \text{otherwise}, \end{cases}$$

where, for simplicity, we consider 0 as an even number. For which values of $c \in \mathbb{R}$ does this define a distribution? (Exactly one answer is correct.)

Hint: You may use the following identity:

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} = \sum_{n=0}^{\infty} \frac{1}{2}\Big(\frac{1}{n!} + \frac{(-1)^n}{n!}\Big) = \frac{1}{2}\Big(\sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\Big).$$

(a) Never.

(b) Only for $c = e^{-1}$.

(c) For all $c \geq 0$.

(d) Only for $c = 2/(e + e^{-1})$.

Solution: (d) is correct. We must have: $1 = \sum_{k=0}^{\infty} \mathbb{P}[X = k]$. The values of $k \in \mathbb{N}_0$ are even precisely when there exists an $n \in \mathbb{N}_0$ such that $k = 2n$. Thus, we obtain:

$$\sum_{k=0}^{\infty} \mathbb{P}[X = k] = \sum_{k=0}^{\infty} \frac{c}{k!}\mathbf{1}_{\{k \text{ is even}\}} = \sum_{n=0}^{\infty} \frac{c}{(2n)!} = \frac{c}{2}\Big(\sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\Big) = \frac{c}{2}\big(e + e^{-1}\big).$$

Solving for c gives $c = 2/(e + e^{-1})$. It is also clear that

$$\frac{2}{(e + e^{-1})k!} \geq 0, \quad \text{for all } k \geq 0.$$

MC 4.2. Let X be a random variable taking values in $\{1, \ldots, 10\}$ with $\mathbb{P}[X = k] = k - c$ for $k \in \{1, \ldots, 10\}$. For which values of $c \in \mathbb{R}$ does this define a distribution? (Exactly one answer is correct.)

(a) If $c = 5.4$.

(b) Never.

(c) If $c = 1$.

(d) If $c = 0$.

Solution: (b) is correct. We have $\sum_{k=1}^{10}(k-c) = \frac{10}{2}(1+10) - 10c = 55 - 10c$. This sum equals 1 if and only if $c = 5.4$. However, for this value, $k - 5.4 < 0$ for every $k \in \{1, \dots, 5\}$. Therefore, the given formula never defines a valid distribution.

Exercise 4.3. [Two independent Poissons] Let X be a random variable taking values in $\mathbb{N}_0$, representing the number of users visiting server A within an hour. We assume that $X \sim \text{Poisson}(\lambda)$ for some $\lambda > 0$.

(a) Recall the definition of the Poisson distribution and argue why it is an appropriate probabilistic model for the described situation.

(b) We know that the server crashes if at least 1000 people visit it within an hour. Express the probability of this event as a function of $\lambda > 0$.

(c) Let Y be a random variable taking values in $\mathbb{N}_0$, representing the number of users visiting server B within an hour. Assume that $Y \sim \text{Poisson}(\gamma)$ for some $\gamma > 0$ and that X and Y are independent. That is,

$$\mathbb{P}[X = i,\, Y = k] = \mathbb{P}[X = i]\mathbb{P}[Y = k]^{(\star)}, \qquad i, k \in \mathbb{N}_0.$$

Find the probabilities $\mathbb{P}[Z = k]$ for $k \in \mathbb{N}_0$, where $Z \coloneqq X + Y$. What is the distribution of the random variable Z?

Hint: It might be helpful to use the binomial formula

$$(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i, \quad \text{for } a, b \in \mathbb{R}, k \in \mathbb{N}_0.$$

$^{(\star)}$ Similarly to the Remark following Definition 2.5 in the lecture notes, one can show that if X and Y take values in $\mathbb{N}_0$, then independence is equivalent to this formula. However, it is crucial that X and Y have **discrete distributions** (i.e., they are supported on a countable set), as this statement does not hold in general.

Solution:

(a) The Poisson distribution is given by

$$\mathbb{P}[X = k] = e^{-\lambda}\frac{\lambda^k}{k!},\ k \in \mathbb{N}_0.$$

The Poisson distribution is a good model for the number of events that occur within a specified time window. It is therefore well-suited to our situation.

(b) We have

$$\mathbb{P}[X \geq 1000] = \sum_{k=1000}^{\infty} \mathbb{P}[X = k] = e^{-\lambda} \sum_{k=1000}^{\infty} \frac{\lambda^k}{k!}.$$

(c) By assumption, we have $\mathbb{P}[X = i,\, Y = k] = \mathbb{P}[X = i]\mathbb{P}[Y = k]$, and for every $\ell \in \mathbb{N}_0$,

$$\{X + Y = k,\, Y = \ell\} = \{X = k - \ell,\, Y = \ell\}.$$

Therefore, using

$$\{Z = k\} = \{X + Y = k\} = \bigcup_{\ell \in \mathbb{N}_0} \{X + Y = k,\, Y = \ell\} = \bigcup_{\ell=0}^{k} \{X = k - \ell,\, Y = \ell\} \quad \text{(disjoint unions)},$$

we obtain

$$\begin{aligned}
\mathbb{P}[Z = k] &= \mathbb{P}[X + Y = k] \\
&= \sum_{\ell=0}^{k} \mathbb{P}[X = k - \ell,\, Y = \ell] = \sum_{\ell=0}^{k} \mathbb{P}[X = k - \ell]\mathbb{P}[Y = \ell] \\
&= \sum_{\ell=0}^{k} e^{-\lambda} \frac{\lambda^{k-\ell}}{(k-\ell)!} e^{-\gamma} \frac{\gamma^{\ell}}{\ell!} \\
&= e^{-(\lambda+\gamma)} \frac{1}{k!} \sum_{\ell=0}^{k} \frac{k!}{\ell!(k-\ell)!} \lambda^{k-\ell} \gamma^{\ell} \\
&= e^{-(\lambda+\gamma)} \frac{(\gamma+\lambda)^k}{k!}.
\end{aligned}$$

We conclude that $Z \sim \text{Poisson}(\lambda + \gamma)$.

Exercise 4.4. [On independence] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let A, B and A_i, $i \in \{1, \dots, n\}$, be events in $\mathcal{F}$. We assume that the A_i are pairwise disjoint.

(a) Show that A and B are independent if and only if A and B^c are independent, which is also equivalent to the independence of A^c and B^c.

(b) Show that if A and each A_i are pairwise independent for all $i \in \{1, \dots, n\}$, then A and $\bigcup_{i=1}^{n} A_i$ are independent.

(c) Assume that $\mathbb{P}[A] = 1$. Show that A and B are independent for all $B \in \mathcal{F}$.

Remark: Try to consider whether the results are intuitive and what their interpretation is.

Solution:

(a) It suffices to show that

$$A, B \text{ are independent} \implies A, B^c \text{ are independent.} \tag{1}$$

Using B^c instead of B, we then obtain via $(B^c)^c = B$ that

$$A, B^c \text{ are independent} \implies A, B \text{ are independent},$$

and swapping A and B, we also get

$$A, B \text{ are independent} \iff A^c, B \text{ are independent.} \tag{2}$$

Replacing B with B^c then gives

$$A, B^c \text{ are independent} \iff A^c, B^c \text{ are independent},$$

and so

$$A, B \text{ are independent} \iff A, B^c \text{ are independent} \iff A^c, B^c \text{ are independent},$$

where the first equivalence follows from (2) by swapping A and B.

To prove (1), we assume

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$$

and show that this implies

$$\mathbb{P}[A \cap B^c] = \mathbb{P}[A]\mathbb{P}[B^c].$$

Since $\Omega = B \cup B^c$ (disjoint), we have $A = A \cap \Omega = (A \cap B) \cup (A \cap B^c)$ (disjoint), leading to $\mathbb{P}[A] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap B^c]$. Hence, $\mathbb{P}[A \cap B^c] = \mathbb{P}[A] - \mathbb{P}[A \cap B]$. Since $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$, we obtain $\mathbb{P}[A \cap B^c] = \mathbb{P}[A](1 - \mathbb{P}[B]) = \mathbb{P}[A]\mathbb{P}[B^c]$, which completes the proof.

(b) By assumption, we have $\mathbb{P}[A \cap A_i] = \mathbb{P}[A]\mathbb{P}[A_i]$ for $i = 1, \dots, n$. Since the events A_i are pairwise disjoint, so are $A \cap A_i$. Thus, we obtain

$$\begin{aligned}\mathbb{P}\Big[A \cap \Big(\bigcup_{i=1}^n A_i\Big)\Big] &= \mathbb{P}\Big[\bigcup_{i=1}^n (A \cap A_i)\Big] = \sum_{i=1}^n \mathbb{P}[A \cap A_i] = \sum_{i=1}^n \mathbb{P}[A]\mathbb{P}[A_i] \\ &= \mathbb{P}[A] \sum_{i=1}^n \mathbb{P}[A_i] = \mathbb{P}[A]\mathbb{P}\Big[\bigcup_{i=1}^n A_i\Big], \text{ as required.}\end{aligned}$$

(c) Using (a), we can show that it suffices to prove the statement for A^c, where $\mathbb{P}[A^c] = 0$. For all events $B \in \mathcal{F}$, we have $A^c \cap B \subseteq A^c$, and thus, using monotonicity, we obtain $\mathbb{P}[A^c \cap B] = 0$. Consequently, $\mathbb{P}[A^c \cap B] = \mathbb{P}[A^c]\mathbb{P}[B]$ holds for every $B \in \mathcal{F}$. From the independence of A^c and B, it follows by (a) that A and B are independent.

Alternatively: If $\mathbb{P}[A] = 1$, then $\mathbb{P}[A^c] = 0$ and hence also $\mathbb{P}[B \cap A^c] = 0$. Since $\Omega = A \cup A^c$ (disjoint) and $\mathbb{P}[A] = 1$, we also get

$$\mathbb{P}[B \cap A] = \mathbb{P}[B \cap A] + \mathbb{P}[B \cap A^c] = \mathbb{P}[B] = \mathbb{P}[B]\mathbb{P}[A].$$

Exercise 4.5. [Studying pays off] We have a representative student of the Probability Theory and Statistics course, whose diligence is represented by a random variable X_1 taking values in $\{0, 1\}$ with distribution $\mathbb{P}[X_1 = 0] = 1 - \mathbb{P}[X_1 = 1] = 4/10$. Here, we interpret $\{X_1 = 1\}$ = "The student studies diligently" and $\{X_1 = 0\}$ = "The student does not study diligently."

Further, we define X_2 taking values in $\{0, 1\}$ as the random variable that represents whether the student attempts to pass the exam, where $\{X_2 = 0\}$ = "The student does not register for the exam" and $\{X_2 = 1\}$ = "The student registers for the exam." We assume that

$$\mathbb{P}[X_2 = 1 | X_1 = x_1] = 1 - \mathbb{P}[X_2 = 0 | X_1 = x_1] = \frac{2 + 3x_1}{5}, \qquad x_1 \in \{0, 1\}.$$

Finally, we define X_3 taking values in $\{0, 1\}$ as the random variable that represents whether the student actually passes the exam, where $\{X_3 = 0\}$ = "The student does not pass the exam" and $\{X_3 = 1\}$ = "The student passes the exam." We assume that

$$\mathbb{P}[X_3 = 1 | X_2 = x_2,\, X_1 = x_1] = \begin{cases} \frac{1+3x_1}{5}, & x_1 \in \{0,1\},\ x_2 = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Describe in words the various conditional probabilities given in the problem, and specify their values.

(b) At the end of the exam session, we know that the student did not pass the exam (i.e., either failed or did not register). What is the conditional probability that the student did not study diligently? That is, compute $\mathbb{P}[X_1 = 0 | X_3 = 0]$.

Solution:

(a) The interpretation and values of the probabilities $\mathbb{P}[X_1 = 0]$ and $\mathbb{P}[X_1 = 1]$ are straightforward.

The value $\mathbb{P}[X_2 = 1 | X_1 = 0] = 2/5$ represents the conditional probability that a student who does not study diligently registers for the exam. Similarly, $\mathbb{P}[X_2 = 1 | X_1 = 1] = 1$ represents the conditional probability that a student who studies diligently registers for the exam. Analogously, $\mathbb{P}[X_2 = 0 | X_1 = 1] = 0$ and $\mathbb{P}[X_2 = 0 | X_1 = 0] = 3/5$ represent the conditional probabilities that a student who does not study diligently, and a student who studies diligently, do not register for the exam, respectively.

The probabilities $\mathbb{P}[X_3 = 1 | X_2 = 1,\, X_1 = 0] = 1/5$ and $\mathbb{P}[X_3 = 1 | X_2 = 1,\, X_1 = 1] = 4/5$ represent the conditional probabilities that a student who does not study diligently but registers for the exam, and a student who studies diligently and registers for the exam, pass the exam. Similarly, $\mathbb{P}[X_3 = 1 | X_2 = 0,\, X_1 = 0] = 0$ and $\mathbb{P}[X_3 = 1 | X_2 = 0,\, X_1 = 1] = 0$ represent the conditional probabilities that a student who does not study diligently and does not register, and a student who studies diligently and does not register, pass the exam. Analogously, we can interpret the probabilities $\mathbb{P}[X_3 = 0 | X_2 = 1,\, X_1 = 0]$, $\mathbb{P}[X_3 = 0 | X_2 = 1,\, X_1 = 1]$, $\mathbb{P}[X_3 = 0 | X_2 = 0,\, X_1 = 0]$, and $\mathbb{P}[X_3 = 0 | X_2 = 0,\, X_1 = 1]$.

(b) We compute

$$\begin{aligned} \mathbb{P}[X_1 = 0 | X_3 = 0] &= \frac{\mathbb{P}[X_1 = 0,\, X_3 = 0]}{\mathbb{P}[X_3 = 0]} = \frac{\sum_{x_2=0}^{1} \mathbb{P}[X_1 = 0,\, X_2 = x_2,\, X_3 = 0]}{\sum_{x_1=0}^{1} \sum_{x_2=0}^{1} \mathbb{P}[X_1 = x_1,\, X_2 = x_2,\, X_3 = 0]} \\ &= \frac{\sum_{x_2=0}^{1} \mathbb{P}[X_3 = 0 | X_1 = 0,\, X_2 = x_2] \mathbb{P}[X_2 = x_2 | X_1 = 0] \mathbb{P}[X_1 = 0]}{\sum_{x_1=0}^{1} \sum_{x_2=0}^{1} \mathbb{P}[X_3 = 0 | X_1 = x_1,\, X_2 = x_2] \mathbb{P}[X_2 = x_2 | X_1 = x_1] \mathbb{P}[X_1 = x_1]} \\ &= \frac{1 \times \frac{3}{5} \times \frac{4}{10} + \frac{4}{5} \times \frac{2}{5} \times \frac{4}{10}}{1 \times \frac{3}{5} \times \frac{4}{10} + \frac{4}{5} \times \frac{2}{5} \times \frac{4}{10} + 0 + \frac{1}{5} \times 1 \times \frac{6}{10}} \\ &= \frac{46}{61} \approx 0.754. \end{aligned}$$

Exercise 4.6. [Funny dice] We have two different dice; on one of them, the 6 is replaced by a 7, and the other one is standard. First, we toss an unfair coin, which shows heads with probability $p \in (0,1)$. If heads occurs, we choose the die with the 6; if tails occurs, we choose the die with the 7. Then, we roll the chosen die twice and compute the sum Y of the obtained face values.

(a) Compute the probability for {the sum of the dice is 10} and {the sum of the dice is 12} as a function of p.

(b) Determine the conditional probabilities of heads given that the sum is 10 and given that the sum 12.

(c) Which of the events {coin toss results in heads}, {sum of the dice is 10}, and {sum of the dice is 12} are independent?

Solution: Let X be the result of the coin toss, where we set $\{X = 0\}$="coin toss results in heads" and $\{X = 1\}$="coin toss results in tails."

(a) For the normal die, there are three outcomes that result in a sum of 10: $(4, 6)$, $(5, 5)$, and $(6, 4)$. For the modified die, there are also three outcomes resulting in 10: $(3, 7)$, $(5, 5)$, and $(7, 3)$. Thus, we have

$$\mathbb{P}[Y = 10|X = 0] = \mathbb{P}[Y = 10|X = 1] = 3 \times \frac{1}{6^2} = \frac{1}{12}.$$

This gives

$$\begin{aligned}\mathbb{P}[Y = 10] &= \mathbb{P}[Y = 10|X = 0]\mathbb{P}[X = 0] + \mathbb{P}[Y = 10|X = 1]\mathbb{P}[X = 1] \\ &= \frac{1}{12} \times p + \frac{1}{12} \times (1 - p) = \frac{1}{12}.\end{aligned}$$

For the normal die, there is only one way to obtain a sum of 12: $(6, 6)$. This gives

$$\mathbb{P}[Y = 12|X = 0] = \frac{1}{6^2}.$$

For the modified die, the two possibilities are $(5, 7)$ and $(7, 5)$, yielding

$$\mathbb{P}[Y = 12|X = 1] = 2 \times \frac{1}{6^2}.$$

Therefore, following the same calculation as before,

$$\begin{aligned}\mathbb{P}[Y = 12] &= \mathbb{P}[Y = 12|X = 0]\mathbb{P}[X = 0] + \mathbb{P}[Y = 12|X = 1]\mathbb{P}[X = 1] \\ &= \frac{1}{36} \times p + \frac{2}{36} \times (1 - p) = \frac{1}{36}(2 - p).\end{aligned}$$

(b) Using the Bayes' rule, we compute

$$\begin{aligned}\mathbb{P}[X = 0|Y = 10] &= \frac{\mathbb{P}[Y = 10|X = 0]\mathbb{P}[X = 0]}{\mathbb{P}[Y = 10|X = 0]\mathbb{P}[X = 0] + \mathbb{P}[Y = 10|X = 1]\mathbb{P}[X = 1]} \\ &= \frac{\mathbb{P}[Y = 10|X = 0]\mathbb{P}[X = 0]}{\mathbb{P}[Y = 10]} = \frac{\frac{3}{36} \times p}{\frac{3}{36}} = p, \\ \mathbb{P}[X = 0|Y = 12] &= \frac{\mathbb{P}[Y = 12|X = 0]\mathbb{P}[X = 0]}{\mathbb{P}[Y = 12|X = 0]\mathbb{P}[X = 0] + \mathbb{P}[Y = 12|X = 1]\mathbb{P}[X = 1]} \\ &= \frac{\mathbb{P}[Y = 12|X = 0]\mathbb{P}[X = 0]}{\mathbb{P}[Y = 12]} = \frac{\frac{1}{36} \times p}{\frac{1}{36}(2 - p)} = \frac{p}{2 - p}.\end{aligned}$$

(c) We compute

$$\begin{aligned}
\mathbb{P}[X=0, Y=10] &= \mathbb{P}[X=0|Y=10]\mathbb{P}[Y=10] = p \times \frac{1}{12} = \mathbb{P}[X=0]\mathbb{P}[Y=10], \\
\mathbb{P}[X=0, Y=12] &= \mathbb{P}[X=0|Y=12]\mathbb{P}[Y=12] = \frac{p}{2-p} \times \frac{1}{36}(2-p) = \frac{p}{36} \\
&\neq p \times \frac{2-p}{36} = \mathbb{P}[X=0]\mathbb{P}[Y=12], \\
\mathbb{P}[Y=10, Y=12] &= 0 \neq \frac{1}{12} \times \frac{2-p}{36} = \mathbb{P}[Y=10]\mathbb{P}[Y=12].
\end{aligned}$$

This shows that only $\{X=0\}$ (“coin toss results in heads”) and $\{Y=10\}$ (“sum of the dice is 10”) are independent.