Probability and Statistics

Exercise sheet 5

MC 5.1. Let X and Y be random variables taking values in $\mathbb{N}$. Which of the following statements are true? (The number of correct answers is between 0 and 6.)

(a) If we know all values of $\mathbb{P}[X = i, Y = j]$ for $i, j \in \mathbb{N}$, then we also know all values of $\mathbb{P}[X = i]$ and $\mathbb{P}[Y = j]$ for $i, j \in \mathbb{N}$.

(b) If we know all values of $\mathbb{P}[X = i]$ and $\mathbb{P}[Y = j]$ for $i, j \in \mathbb{N}$, then we also know all values of $\mathbb{P}[X = i, Y = j]$ for $i, j \in \mathbb{N}$.

(c) Statement (b) is true, provided that we know that X and Y are independent.

(d) Statement (b) is true, provided that we know that $Z_1 := e^{-X}$ and $Z_2 := \sqrt{Y}$ are independent.

(e) Statement (b) is true, provided that we know that $Z_1 := e^{-X}$ and $Z_2 := Y \bmod 2$ are independent.

(f) If we assume $\mathbb{P}[Y = j] > 0$ and we know all values of $\mathbb{P}[X = i \mid Y = j]$ and $\mathbb{P}[Y = j]$ for $i, j \in \mathbb{N}$, then we also know all values of $\mathbb{P}[X = i, Y = j]$ for $i, j \in \mathbb{N}$.

Exercise 5.2. Let $N \in \mathbb{N}$. Consider the set

$$\Omega = \{0,1\}^N = \big\{(a_1, \dots, a_N) : a_1, \dots, a_N \in \{0,1\}\big\}$$

and the σ-algebra $\mathcal{A} = 2^\Omega$, as well as the probability measure $\mathbb{P}$ defined by

$$\mathbb{P}\big[\{(a_1, \dots, a_N)\}\big] = \begin{cases} \frac{1}{2^{N-1}}, & \text{if } a_1 + \cdots + a_N \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $X_1, \dots, X_N$ be random variables defined by $X_i((a_1, \dots, a_N)) = a_i$, $(a_1, \dots, a_N) \in \Omega$, $i \in \{1, \dots, N\}$.

(a) Show that $X_1, \dots, X_{N-1}$ are independent.

(b) Show that $X_1, \dots, X_N$ are not independent.

Exercise 5.3. Let X be a random variable with the cumulative distribution function

$$F_X(a) = \begin{cases} 0, & \text{if } a < 1, \\ 1/5, & \text{if } 1 \le a < 4, \\ 3/4, & \text{if } 4 \le a < 6, \\ 1, & \text{if } 6 \le a. \end{cases}$$

(a) Sketch the cumulative distribution function of X.

(b) Find the distribution of X (i.e. the values of $p(x) = \mathbb{P}[X = x]$) and sketch it.

(c) Compute the probabilities $\mathbb{P}[X = 6], \mathbb{P}[X = 5], \mathbb{P}[2 < X < 5.5], \mathbb{P}[0 \le X < 4]$.

Exercise 5.4. Let X and Y be two discrete random variables with the following joint distribution:

$$p(j,k) = \mathbb{P}[X = j, Y = k] = \begin{cases} C\left(\frac{1}{2}\right)^k & \text{for } k = 2, 3, \dots \text{ and } j = 1, 2, \dots, k-1 \\ 0 & \text{otherwise,} \end{cases}$$

for some constant $C \in \mathbb{R}$.

(a) Determine the constant C.

(b) Compute the (marginal) distributions p_X and p_Y of X and Y.

(c) Are X and Y independent?

Exercise 5.5. [Monty Hall Problem] You are in a game show and have a choice between three doors. Behind one door is a car, while the other two doors hide goats. You select a door, and the host, who knows what is behind the doors, opens another door revealing a goat. The host then asks you: "Would you like to stick with your originally chosen door or switch to another one?" Assuming you prefer cars over goats, what should you do?

(a) Construct an appropriate model to answer this question using conditional probabilities.

(b) Try to find an alternative solution (which must, of course, yield the same answer).