Probability and Statistics

Exercise sheet 6

MC 6.1. Let X be a random variable with distribution function

$$F(a) = \begin{cases} 0, & a < 0, \\ \frac{a}{2}, & 0 \leq a < 1, \\ \frac{2}{3}, & 1 \leq a < 2, \\ \frac{a+1}{4}, & 2 \leq a < 3, \\ 1, & a \geq 3. \end{cases}$$

Does X have a density? (Exactly one answer is correct.)

(a) Yes.

(b) No.

MC 6.2. Let

$$F(a) = \begin{cases} 0, & a < 0, \\ \frac{a}{2}, & 0 \leq a < 1, \\ \frac{a+1}{4}, & 1 \leq a < 3, \\ 1, & a \geq 3 \end{cases}$$

be a distribution function. Which of the following statements is correct? (Exactly one answer is correct.)

(a) F has no density.

(b) F has a density given by

$$f_{(b)}(a) = \begin{cases} 0, & a < 0, \\ \frac{1}{2}, & 0 \leq a < 1, \\ \frac{1}{4}, & 1 \leq a < 3, \\ 0, & a \geq 3. \end{cases}$$

(c) F has a density given by

$$f_{(c)}(a) = \begin{cases} 0, & a < 0, \\ \frac{1}{2}, & 0 \leq a < 1, \\ \frac{2}{4}, & 1 \leq a < 3, \\ 1, & a \geq 3. \end{cases}$$

(d) We cannot determine whether F has a density.

MC 6.3. Let F be a distribution function and f the corresponding density. Which of the following statements are true? (The number of correct answers is between 0 and 4.)

(a) $f \geq 0$, but not necessarily $F \geq 0$.

(b) f is right-continuous.

(c) $f \leq 1$.

(d) F can be discontinuous, but only at finitely many points.

Exercise 6.4. Let T be a random variable with distribution function

$$F_T(a) = \begin{cases} 0, & \text{if } a < 0, \\ 1 - e^{-2a}, & \text{if } a \geq 0. \end{cases}$$

(a) Sketch the distribution function.

(b) Show that T is a continuous random variable.

(c) Compute the density function of T.

(d) Compute the probabilities $\mathbb{P}[T = 2], \mathbb{P}[T \leq 1], \mathbb{P}[T \geq 2], \mathbb{P}[1 < T < 4]$.

Exercise 6.5. We consider a sensor placed at a volcano crater that is monitoring for a potential eruption. Starting from the beginning of the measurements, we assume that the sensor fails within one minute with probability $\frac{1}{20}$ due to excessive damage. Let the random variable Y denote the lifetime of the sensor in minutes. We assume that it holds $Y \sim \text{Exp}(\lambda)$, i.e., Y is exponentially distributed with parameter $\lambda > 0$.

(a) Determine the value of λ.

Hint: The correct result is $\lambda = -\log(0.95)$, which you can use for the remaining parts.

(b) What is the probability that the sensor survives more than 10 minutes?

(c) Given that the sensor has already survived more than 20 minutes, what is the conditional probability that it will survive another 10 minutes?

Exercise 6.6. Assume that $-\infty < a < b < \infty$ and $c > 0$.

(a) Let $U \sim \mathcal{U}([0, 1])$. Find the density of the random variable $U' := a + (b - a)U$.

(b) Let $T \sim \text{Exp}(\lambda)$ with parameter $\lambda > 0$. Find the density of $T' := cT^2$.