Probability and Statistics

Exercise sheet 6 - Solutions

MC 6.1. Let X be a random variable with distribution function

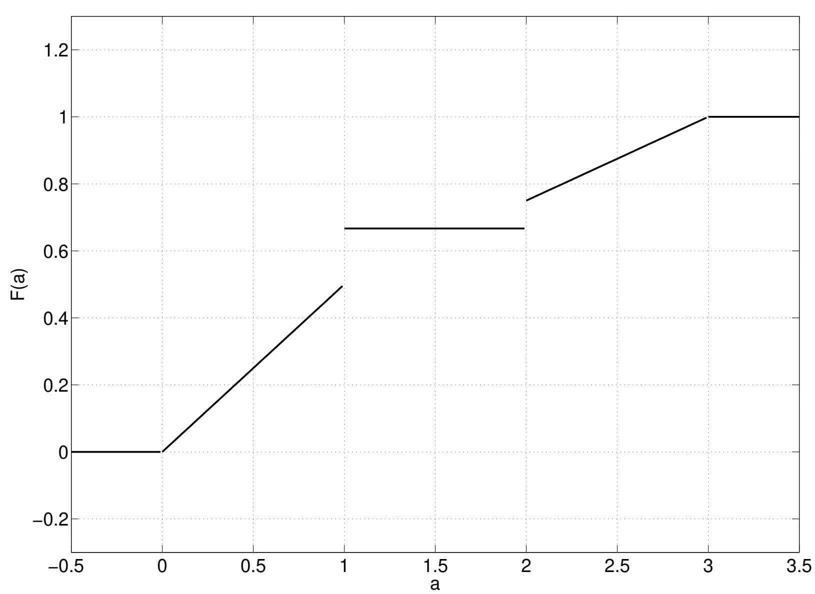
$$F(a) = \begin{cases} 0, & a < 0, \\ \frac{a}{2}, & 0 \le a < 1, \\ \frac{2}{3}, & 1 \le a < 2, \\ \frac{a+1}{4}, & 2 \le a < 3, \\ 1, & a \ge 3. \end{cases}$$

Does X have a density? (Exactly one answer is correct.)

(a) Yes.

(b) No.

Solution: The graph of F is as follows:



Thus, X cannot have a density function, since the cumulative distribution function F has jumps, i.e., it is not continuous. Observe that F is differentiable at all but finitely many points. However, because it is not continuous, it cannot have a density. For instance, we see that $\mathbb{P}[X = 1] = 2/3 - 1/2 = 1/6 > 0$, which contradicts the existence of a density. The correct answer is thus (b).

MC 6.2. Let

$$F(a) = \begin{cases} 0, & a < 0, \\ \frac{a}{2}, & 0 \leq a < 1, \\ \frac{a+1}{4}, & 1 \leq a < 3, \\ 1, & a \geq 3 \end{cases}$$

be a distribution function. Which of the following statements is correct? (Exactly one answer is correct.)

(a) F has no density.

(b) F has a density given by

$$f_{(b)}(a) = \begin{cases} 0, & a < 0, \\ \frac{1}{2}, & 0 \leq a < 1, \\ \frac{1}{4}, & 1 \leq a < 3, \\ 0, & a \geq 3. \end{cases}$$

(c) F has a density given by

$$f_{(c)}(a) = \begin{cases} 0, & a < 0, \\ \frac{1}{2}, & 0 \leq a < 1, \\ \frac{2}{4}, & 1 \leq a < 3, \\ 1, & a \geq 3. \end{cases}$$

(d) We cannot determine whether F has a density.

Solution: We can compute the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}a} F(a) = \begin{cases} 0, & a < 0, \\ \frac{1}{2}, & 0 < a < 1, \\ \frac{1}{4}, & 1 < a < 3, \\ 0, & a > 3, \\ \text{does not exist}, & a \in \{0, 1, 3\}. \end{cases} \tag{1}$$

Therefore, $f_{(b)}$ is a candidate for a density function. One can verify by direct computation that $F(a) = \int_{-\infty}^{a} f_{(b)}(x)\mathrm{d}x$ for all $a \in \mathbb{R}$, and hence (b) is correct.

Alternatively, we can observe that F is continuous and piecewise continuously differentiable. It then follows that F has a density given by the derivative of F.

Note: Observe that the value of a density can be changed at finitely many points without affecting the value of its integral. Thus, it is not a problem that the derivative in (1) does not exist at $a = 0$, $a = 1$, and $a = 3$.

Optional technical remark: Let f be a density of F. Then g is also a density of F if and only if "$f = g$ almost everywhere with respect to the Lebesgue measure" holds. In such a case, we necessarily have

$$\int_{-\infty}^{a} f(x)\mathrm{d}x = \int_{-\infty}^{a} g(x)\mathrm{d}x \quad \text{for all } a \in \mathbb{R}.$$

For example, if $f(x) = g(x)$ for all but finitely many points, then $f = g$ almost everywhere with respect to the Lebesgue measure.

MC 6.3. Let F be a distribution function and f the corresponding density. Which of the following statements are true? (The number of correct answers is between 0 and 4.)

(a) $f \geq 0$, but not necessarily $F \geq 0$.

(b) f is right-continuous.

(c) $f \leq 1$.

(d) F can be discontinuous, but only at finitely many points.

Solution: None of the answers are correct. (a) is false because $F \geq 0$ always holds. (b) and (c) are true for F, but not necessarily for f. Finally, (d) is false because F cannot be discontinuous if it has a density.

Exercise 6.4. Let T be a random variable with distribution function

$$F_T(a) = \begin{cases} 0, & \text{if } a < 0, \\ 1 - e^{-2a}, & \text{if } a \geq 0. \end{cases}$$

(a) Sketch the distribution function.

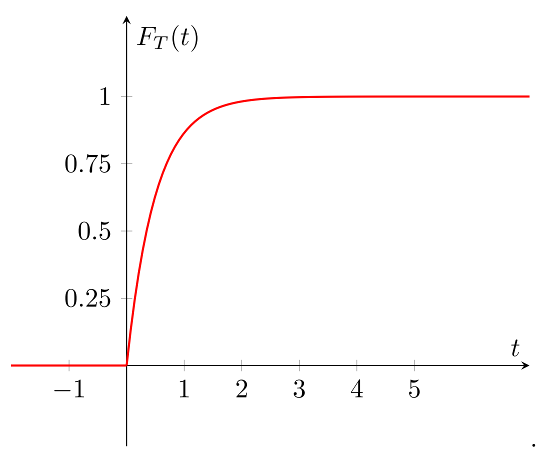
(b) Show that T is a continuous random variable.

(c) Compute the density function of T.

(d) Compute the probabilities $\mathbb{P}[T = 2], \mathbb{P}[T \leq 1], \mathbb{P}[T \geq 2], \mathbb{P}[1 < T < 4]$.

Solution:

(a) The following sketch shows the distribution function of T:



(b) We observe that the distribution function F_T is continuous and piecewise continuously differentiable. More precisely, F_T is continuous and continuously differentiable on $(-\infty, 0)$ and on

$(0, \infty)$. Therefore, T is a continuous random variable.

(c) Since F_T is continuous and piecewise continuously differentiable, we can compute the derivative of F_T to get the density. We obtain

$$f_T(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_T(t) = \begin{cases} 0 & \text{if } t < 0, \\ 2e^{-2t} & \text{if } t \geq 0, \end{cases}$$

where we have arbitrarily defined the value at $t = 0$.

(d) Since the random variable T is continuous, we have $\mathbb{P}[T = t] = 0$ for all $t \in \mathbb{R}$. Therefore, in particular,

$$\mathbb{P}[T = 2] = 0.$$

Furthermore, we have

$$\mathbb{P}[T \leq 1] = F_T(1) = 1 - e^{-2},$$
$$\mathbb{P}[T \geq 2] = 1 - \mathbb{P}[T < 2] = 1 + \underbrace{\mathbb{P}[T = 2]}_{=0} - \underbrace{\mathbb{P}[T \leq 2]}_{=F_T(2)} = e^{-4},$$
$$\mathbb{P}[1 < T < 4] = \mathbb{P}[T < 4] - \mathbb{P}[T \leq 1] = \underbrace{\mathbb{P}[T \leq 4]}_{=F_T(4)} - \underbrace{\mathbb{P}[T = 4]}_{=0} - \underbrace{\mathbb{P}[T \leq 1]}_{=F_T(1)} = e^{-2} - e^{-8}.$$

Exercise 6.5. We consider a sensor placed at a volcano crater that is monitoring for a potential eruption. Starting from the beginning of the measurements, we assume that the sensor fails within one minute with probability $\frac{1}{20}$ due to excessive damage. Let the random variable Y denote the lifetime of the sensor in minutes. We assume that it holds $Y \sim \mathrm{Exp}(\lambda)$, i.e., Y is exponentially distributed with parameter $\lambda > 0$.

(a) Determine the value of λ.

Hint: The correct result is $\lambda = -\log(0.95)$, which you can use for the remaining parts.

(b) What is the probability that the sensor survives more than 10 minutes?

(c) Given that the sensor has already survived more than 20 minutes, what is the conditional probability that it will survive another 10 minutes?

Solution:

(a) According to the assumption in the problem, we have $\mathbb{P}[Y \leq 1] = \frac{1}{20} = 0.05$, hence $\mathbb{P}[Y > 1] = 0.95$. From the lecture, we know that $\mathbb{P}[T > t] = e^{-\lambda t}$ for a random variable T that is exponentially distributed with parameter λ. Therefore, we must have

$$0.95 = e^{-\lambda},$$

which gives $\lambda = -\log(0.95)$.

(b) The probability that the sensor survives more than 10 minutes is

$$\mathbb{P}[Y > 10] = e^{-\lambda \cdot 10} = e^{10 \log(0.95)} = 0.95^{10} \approx 0.5987.$$

Alternatively,

$$\mathbb{P}[Y > 10] = 1 - \mathbb{P}[Y \leq 10] = 1 - F_Y(10) = 1 - (1 - e^{\log(0.95)\cdot 10}) = 0.95^{10}.$$

(c) We know that the exponential distribution is memoryless, i.e., for all $s, t \geq 0$ it holds that

$$\mathbb{P}[T > t + s | T > t] = \frac{\mathbb{P}[T > t+s, T > t]}{\mathbb{P}[T > t]} = \frac{\mathbb{P}[T > t+s]}{\mathbb{P}[T > t]} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbb{P}[T > s].$$

Thus, we find that

$$\mathbb{P}[Y > 30 | Y > 20] = \mathbb{P}[Y > 10] = 0.95^{10}.$$

Exercise 6.6. Assume that $-\infty < a < b < \infty$ and $c > 0$.

(a) Let $U \sim \mathcal{U}([0,1])$. Find the density of the random variable $U' := a + (b-a)U$.

(b) Let $T \sim \mathrm{Exp}(\lambda)$ with parameter $\lambda > 0$. Find the density of $T' := cT^2$.

Solution:

(a) For $x \in \mathbb{R}$ we have that

$$F_{U'}(x) = \mathbb{P}[U' \leq x] = \mathbb{P}\left[U \leq \frac{x-a}{b-a}\right] = \begin{cases} 0 & \text{if } x < a, \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b, \\ 1 & \text{if } x > b. \end{cases}$$

Thus, $U' \sim \mathcal{U}([a,b])$, and the density is given by

$$f_{U'}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{U'}(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{else.} \end{cases}$$

(b) Since $T' \geq 0$ $\mathbb{P}$–a.s., we have $F_{T'}(x) = 0$ for $x < 0$. For $x \geq 0$:

$$F_{T'}(x) = \mathbb{P}[T' \leq x] = \mathbb{P}[cT^2 \leq x] = \mathbb{P}\left[T \leq \sqrt{\frac{x}{c}}\right] = F_T\left(\sqrt{\frac{x}{c}}\right) = 1 - \exp\left(-\lambda\sqrt{\frac{x}{c}}\right).$$

Hence, the density is:

$$f_{T'}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{T'}(x) = \begin{cases} \frac{\lambda}{2\sqrt{cx}} \exp\left(-\lambda\sqrt{\frac{x}{c}}\right) & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Alternatively, we have for $x \geq 0$:

$$\mathbb{P}[T' \leq x] = \mathbb{P}\left[T \leq \sqrt{\frac{x}{c}}\right] = \int_0^{\sqrt{x/c}} f_T(t)\mathrm{d}t = \int_0^x f_T\left(\sqrt{\frac{y}{c}}\right)\frac{1}{2\sqrt{cy}}\mathrm{d}y,$$

where we have used the substitution $y = ct^2$ in the last step. We thus obtain that

$$f_{T'}(y) = \begin{cases} f_T\left(\sqrt{\frac{y}{c}}\right)\frac{1}{2\sqrt{cy}}, & y \geq 0 \\ 0, & y < 0, \end{cases}$$

is the density of T' and recover the same result as above.