Probability and Statistics

Exercise sheet 7

MC 7.1. Let Z be a random variable with distribution function:

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ 0.1, & \text{if } 0 \leq z < 1, \\ 0.5, & \text{if } 1 \leq z < 3, \\ 0.8, & \text{if } 3 \leq z < 5, \\ 1, & \text{if } z \geq 5. \end{cases}$$

(Exactly one answer is correct in each question.)

1. Is $\mathbb{E}[Z] \geq 3$?
   (a) Yes.
   (b) No.
2. Is $\mathbb{P}[Z \leq 3] = \mathbb{P}[Z \geq 3]$?
   (a) Yes.
   (b) No.
3. Is $\mathbb{P}[3.5 \leq Z \leq 5.5] = 0.2$?
   (a) Yes.
   (b) No.
4. What is $\mathbb{E}[Z^2]$?
   (a) $\mathbb{E}[Z^2] = 8.1$.
   (b) $\mathbb{E}[Z^2] = 3.5$.
   (c) $\mathbb{E}[Z^2] = 21$.
   (d) Doesn't exist.
   (e) $\mathbb{E}[Z^2] = \infty$.
5. Is $\mathbb{P}[Z = 0] = 0$?
   (a) Yes.
   (b) No.

Exercise 7.2. Let $r > 1$ and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{for } x \leq 1, \\ cx^{-r} & \text{for } x > 1, \end{cases}$$

for some constant $c \in \mathbb{R}$.

(a) Determine the constant c such that f is a density.

(b) Let X be a random variable with density $f_X = f$. Compute the cumulative distribution function of X.

(c) Compute the expected value of X. For which values of r is the expected value finite?

Exercise 7.3. $k \in \mathbb{N}$ hunters each shoot once simultaneously at a flock of $m \in \mathbb{N}$ ducks. They independently choose which duck to aim at, and they hit their chosen duck independently of each other and independently of the duck selected, with probability $p \in (0, 1)$.

Introduce for each duck $n \in \{1, \dots, m\}$ a random variable X_n indicating whether the duck was hit (by at least one hunter) or not. We define $\{X_n = 1\}$ = "n-th duck not hit" and $\{X_n = 0\}$ = "n-th duck hit".

(a) What is the distribution of X_n for $n = 1, \dots, m$?

(b) What is the expected number of unharmed ducks?

(c) Are the events $\{X_n = 0\}$, $n \in \{1, \dots, m\}$ independent? Consider only the case $k < m$ for simplicity.

Exercise 7.4. We consider a circle which has a random radius R. The radius R is exponentially distributed with expectation $1/\lambda$ for some $\lambda > 0$. Let us denote by A the (random) area of this circle. Determine:

(a) The distribution function and the density function of A;

(b) The expected value of A.

Exercise 7.5. A random variable X has the density function:

$$f(x) = \begin{cases} \dfrac{c}{(1+x)^5}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

(a) Find the value of c and the distribution function of X.

(b) Find $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

Hint: It might be easier to first compute $\mathbb{E}[1 + X]$ and $\mathbb{E}[(1 + X)^2]$ and then use linearity.

(c) What are the distribution function and the density of $Y := e^X$?