Probability and Statistics

Exercise sheet 8

MC 8.1. Let X and Y be two independent and identically distributed random variables taking values in $\{1, 2\}$, such that

$$\mathbb{P}[X = i] = \mathbb{P}[Y = i] = \frac{1}{2}, \quad i \in \{1, 2\}.$$

Define the random variable

$$Z \coloneqq X + Y.$$

(Exactly one answer is correct for each question.)

1. What is the cumulative distribution function of Z?

   (a) $F_Z(a) = \begin{cases} 0, & a < 2, \\ \frac{1}{4}, & 2 \le a < 3, \\ \frac{3}{4}, & 3 \le a < 4, \\ 1, & 4 \le a. \end{cases}$

   (b) $F_Z(a) = \begin{cases} 0, & a < 2, \\ \frac{1}{4}, & 2 \le a < 3, \\ \frac{1}{2}, & 3 \le a < 4, \\ 1, & 4 \le a. \end{cases}$

   (c) $F_Z(a) = \begin{cases} 0, & a < 2, \\ \frac{1}{3}, & 2 \le a < 3, \\ \frac{3}{4}, & 3 \le a < 4, \\ 1, & 4 \le a. \end{cases}$

   (d) $F_Z(a) = \begin{cases} 0, & a < 2, \\ \frac{1}{4}, & 2 \le a < 3, \\ \frac{5}{6}, & 3 \le a < 4, \\ 1, & 4 \le a. \end{cases}$

2. What is the value of $\mathrm{Cov}(X, Z)$?

   (a) $\mathrm{Cov}(X, Z) = \frac{1}{4}$.
   (b) $\mathrm{Cov}(X, Z) = 0$.
   (c) $\mathrm{Cov}(X, Z) = \frac{1}{2}$.
   (d) $\mathrm{Cov}(X, Z) = \frac{19}{4}$.

MC 8.2. Let X and Y be random variable with joint density

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{9}, & 1 \le x \le 4 \text{ and } 1 \le y \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

(Exactly one answer is correct for each question.)

1. Are X and Y identically distributed, i.e., do X and Y have the same distribution?
   (a) Yes.
   (b) No.
2. Are X and Y independent?
   (a) Yes.
   (b) No.
3. Are X and Y i.i.d.?
   (a) Yes.
   (b) No.
4. Which of the following functions is the density function f_X of X?
   (a) $x \mapsto 1$ for $x \in \mathbb{R}$.
   (b) $x \mapsto \frac{1}{9}$ for $x \in \mathbb{R}$.
   (c) $x \mapsto \frac{1}{3}$ for $x \in \mathbb{R}$.
   (d) $x \mapsto \begin{cases} \frac{x}{9}, & \text{if } x \in [1,4], \\ 1, & \text{if } x > 4, \\ 0, & \text{otherwise.} \end{cases}$
   (e) $x \mapsto \begin{cases} \frac{x-1}{3}, & \text{if } x \in [1,4], \\ 1, & \text{if } x > 4, \\ 0, & \text{otherwise.} \end{cases}$
   (f) $x \mapsto \begin{cases} \frac{1}{9}, & \text{if } x \in [1,4], \\ 0, & \text{otherwise.} \end{cases}$
   (g) $x \mapsto \begin{cases} \frac{1}{3}, & \text{if } x \in [1,4], \\ 0, & \text{otherwise.} \end{cases}$
   (h) $x \mapsto \begin{cases} \frac{x}{9}, & \text{if } x \in [1,4], \\ 0, & \text{otherwise.} \end{cases}$
5. Which of the functions from Question 4 is the distribution function F_X of X?

MC 8.3. Let X and Y be two random variables with $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$. Which of the following statements is generally true? (Exactly one answer is correct.)

(a) $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.

(b) If X and Y are independent, then $\mathrm{Var}[X-Y] = \mathrm{Var}[X] - \mathrm{Var}[Y]$.

(c) $\mathrm{Var}[X] = \mathrm{Var}[-X]$.

(d) The equality $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ is only true if X and Y are independent.

MC 8.4. Let X be a random variable with $\mathbb{E}[X^2] < \infty$. Which of the following statements are true? (The number of correct answers is between 0 and 4.)

(a) $\mathbb{E}[X^2] = (\mathbb{E}[X])^2$.

(b) $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$.

(c) If X is centered (i.e., $\mathbb{E}[X] = 0$), then $\mathrm{Var}[X] = \mathbb{E}[X^2]$.

(d) $\mathrm{Var}[X] > 0$.

(e) The random variable $Y \coloneqq X - \mathbb{E}[X]$ has the same variance as X.

MC 8.5. Let X be a random variable that takes values in the set $\{0, 1, 3\}$ with $\mathbb{E}[X] = 2$. Which of the following statements are true? (The number of correct answers is between 0 and 4.)

(a) $\mathbb{P}[X = 0] \geq \frac{1}{3}$.

(b) $\mathbb{P}[X = 1] \geq \frac{1}{2}$.

(c) $\mathbb{P}[X = 0] \leq \frac{1}{6}$.

(d) $\mathbb{P}[X = 3] \geq \frac{1}{2}$.

Exercise 8.6. Based on many years of research, it is known that the lead concentration X in a soil sample is approximately normally distributed. It is also known that the expected value is 32 ppb (parts per billion) and that the standard deviation is 6 ppb. (The standard deviation is defined as the square root of the variance, i.e. $\mathrm{sd}(X) \coloneqq \sqrt{\mathrm{Var}[X]}$.)

(a) Sketch the density of X and indicate in the sketch the probability that a soil sample contains between 26 and 38 ppb of lead.

(b) What is the probability that a soil sample contains at most 40 ppb of lead?

Hint: Standardize the random variable and use the table of the standard normal distribution below.

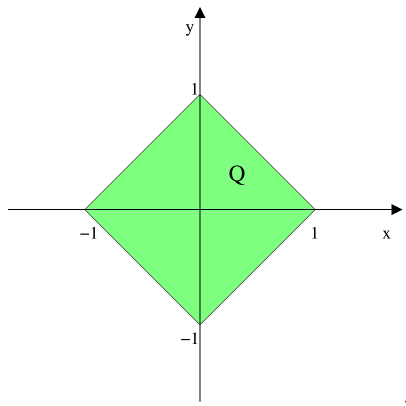
(c) What is the probability that a soil sample contains at most 27 ppb of lead?

(d) What lead concentration is not exceeded with 97.5% probability? That is, find the value c such that the probability that the lead concentration is less than or equal to c is exactly 97.5%.

(e) What lead concentration is not exceeded with 10% probability?

(f) What is the value of the probability indicated in part (a)?

Exercise 8.7. The joint density $f(x, y)$ of two continuous random variables X and Y is constant on the square Q (see sketch) and zero outside of Q.



(a) Determine the joint density of (X, Y).

(b) Determine the marginal densities f_X and f_Y of the random variables X and Y.

(c) Are X and Y independent?

(d) What is the answer to (c) if the square Q is rotated by 45 degrees?

Exercise 8.8. For two independent random variables X and Y, it is known from the lecture that

$$\mathrm{Cov}(X, Y) = 0,$$

i.e., the random variables are uncorrelated. In this problem, we show that the converse is not true in general.

(a) Let $X \sim \mathcal{U}([-\pi, \pi])$. Show that $Y := \cos(X)$ and $Z := \sin(X)$ are uncorrelated, i.e.,

$$\mathrm{Cov}(Y, Z) = 0.$$

(b) Show that Y and Z are not independent.

Hint: If Y and Z were independent, then Y^2 and Z^2 would also be independent. Disprove the latter by considering $\mathbb{P}[Y^2 \leq 1/2,\ Z^2 \leq 1/2]$.

Exercise 8.9. Let X and Y be two random variables that can only take the values 0 and 1. The joint distribution of (X, Y) satisfies:

$$\mathbb{P}[X = 0] = \frac{1}{2}, \quad \mathbb{P}[Y = 0] = \frac{1}{3}, \quad \text{and} \quad \mathbb{P}[X = 0, Y = 0] = p.$$

(a) What values can p take, and for which values of p are X and Y independent?

(b) Compute the expectations $\mathbb{E}[X]$, $\mathbb{E}[Y]$, and $\mathbb{E}[XY]$, as well as the variances $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$, as functions of p. When does $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ hold?

(c) Give an example of random variables U and V such that $\mathbb{E}[UV] = \mathbb{E}[U]\mathbb{E}[V]$, but U and V are not independent.

Quantile table for the standard normal distribution

0.5	0.75	0.9	0.95	0.975	0.99	0.995	0.999
0	0.6745	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902

.

For instance, $\Phi^{-1}(0.9) = 1.2816$, where Φ is the distribution function of $\mathcal{N}(0,1)$.

Table of standard normal distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

For instance, $\mathbb{P}[Z \leq 1.96] = 0.975$.