Probability and Statistics

Exercise sheet 9

MC 9.1. Let $X \sim \mathcal{N}(0,1)$ and let Φ denote the distribution function of X. Which of the following statements are true? (The number of correct answers is between 0 and 4.)

(a) For every $a < b$, we have $\mathbb{P}[X \in (a,b]] = \Phi(b) - \Phi(a)$.

(b) The random variable $Z := 2X - 3$ has distribution $\mathcal{N}(-3, 2)$.

(c) For every $a < b$, we have $\mathbb{P}[X \in (a,b]] > 0$.

(d) $\mathbb{P}[X \leq 34] = 1 - \Phi(-34)$.

Exercise 9.2. Let X and Y be random variables with joint density

$$f_{X,Y}(x,y) = cxye^{-y}\mathbf{1}_{[0,y]}(x)\mathbf{1}_{[0,\infty)}(y) = \begin{cases} cxye^{-y}, & \text{if } 0 \leq x \leq y \text{ and } 0 \leq y, \\ 0, & \text{otherwise,} \end{cases}$$

for a constant $c \in \mathbb{R}$.

(a) Find the value of c.

Hint: You may use the identity

$$\int_0^\infty y^n e^{-y} \mathrm{d}y = n! \quad \text{for } n \in \mathbb{N}.$$

(b) Find the marginal density f_Y of Y.

(c) Compute the expectation $\mathbb{E}[X^2/Y]$.

Exercise 9.3. Let $S \sim \mathcal{N}(-5, 4^2)$ and $T \sim \mathcal{N}(10, 3^2)$ be independent.

(a) Compute $\mathbb{P}[S < T]$.

(b) Would the computation of $\mathbb{P}[S < T]$ also be correct without the assumption of independence?

(c) Compute the variance $\mathrm{Var}[R]$ of $R := S - 2T$.

(d) Would the computation of $\mathrm{Var}[R]$ also be correct without the assumption of independence?

Hint: You may use the following fact: If X and Y are **independent** and normally distributed, then $X + Y$ is also normally distributed.

Let further $U \sim \mathcal{U}([1,3])$ and $V \sim \mathcal{U}([0,4])$ (i.e., $f_U(u) = \frac{1}{2}\mathbf{1}_{[1,3]}(u)$ and $f_V(v) = \frac{1}{4}\mathbf{1}_{[0,4]}(v)$) be independent.

(e) Compute $\mathbb{E}[2U + V^3]$.

(f) Would the computation of $\mathbb{E}[2U + V^3]$ also be correct without the assumption of independence?

Exercise 9.4. A rectangle is given with random side lengths X and Y. The joint density of X and Y is given by

$$f_{X,Y}(x,y) := \begin{cases} C(x^2+y^2), & 0 \le x \le 1, 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the constant C.

(b) Compute the marginal densities of X and Y.

(c) Are X and Y independent? Justify your answer.

(d) Compute the probability that side X is more than twice as long as side Y.

(e) Compute the expected area of the rectangle.

Exercise 9.5. Let $X_1, X_2, \dots$ be a sequence of independent, identically distributed random variables with distribution function F. The empirical distribution function $F_n : \mathbb{R} \times \Omega \to [0,1]$ is defined as

$$F_n(t,\omega) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{X_i(\omega)\le t\}}.$$

The value $F_n(t,\omega)$ thus describes the relative frequency of those $X_i(\omega)$ that are less that t among the first n. Therefore, $\omega \mapsto F_n(t,\omega)$ is a random variable for each $t \in \mathbb{R}$.

(a) Let $t \in \mathbb{R}$ be arbitrary and define $Y_i := \mathbf{1}_{\{X_i \le t\}}$ for $i \in \mathbb{N}$. Show that $Y_1, Y_2, \dots$ is a sequence of independent, identically distributed random variables. What is $\mathbb{E}[Y_1]$?

(b) Show that for every $t \in \mathbb{R}$, the empirical distribution function $F_n(t)$ converges almost surely to $F(t)$ as $n \to \infty$.

Exercise 9.6. In this problem, we compute the limit

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

(a) Let $X \sim \text{Poisson}(\lambda)$ for some $\lambda > 0$. Compute $\mathbb{E}[X]$ and $\text{Var}[X]$.
Hint: It might be easier to compute $\mathbb{E}[X(X-1)]$ and then use linearity to find $\mathbb{E}[X^2]$.

(b) Use the central limit theorem to show that the limit above is equal to $1/2$.
Hint: Recall that for two independent random variables X, Y with $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$, and $Y \sim \text{Poisson}(\mu)$, $\mu > 0$, the sum satisfies $X + Y \sim \text{Poisson}(\lambda + \mu)$.

Table of standard normal distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

For instance, $\mathbb{P}[Z \leq 1.96] = 0.975$.