# Problems and suggested solution

# Problem 1

Let $n \in \mathbb{N}$, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, $A \in \mathrm{GL}_n(\mathbb{Z})$. Define $T_A \colon \mathbb{T}^n \to \mathbb{T}^n$ by

$$\forall v \in \mathbb{R}^n \quad T_A(v + \mathbb{Z}^n) = Av + \mathbb{Z}^n.$$

Show that the following are equivalent.

1. $A$ does not have a root of unity as an eigenvalue.
2. $T_A$ is ergodic for the Haar measure on $\mathbb{T}^n$.
3. $T_A$ is mixing for the Haar measure on $\mathbb{T}^n$.

**Solution:**

We saw in class that in general mixing implies ergodicity, so (3) implies (2).
We start with (1) implies (3). We saw in class that mixing is equivalent to the statement that for $f, g \in \mathrm{L}^2(\mathbb{T}^n)$,

$$\langle f \circ T_A^n, g\rangle \overset{n\to\infty}{\longrightarrow} \int_{\mathbb{T}^n} f \mathrm{d}m \int_{\mathbb{T}^n} \overline{g} \mathrm{d}m, \qquad (\star)$$

where $m$ is the normalized Haar measure on $\mathbb{T}^n$. We have also seen in class that it suffices to verify this property on a dense subset of $\mathrm{L}^2(\mathbb{T}^n)$, i.e., for any dense subset $E \subseteq \mathrm{L}^2(\mathbb{T}^n)$, if $(\star)$ holds for any two elements $f, g \in E$, then $T_A$ is mixing.
Given $k \in \mathbb{Z}^n$, let

$$\chi_k \colon \mathbb{T}^n \longrightarrow \mathbb{C}, \quad x \longmapsto \exp\Big(2\pi \mathrm{i}\langle k, x\rangle\Big).$$

Using Fourier series, we know that $V = \mathrm{span}_{\mathbb{C}}\{\chi_k \colon k \in \mathbb{Z}^n\}$ is dense in $\mathrm{L}^2(\mathbb{T}^n)$. Let $\varphi = \sum_{k\in\mathbb{Z}^n} a_k\chi_k$ and $\psi = \sum_{k\in\mathbb{Z}^n} b_k\chi_k$ be elements in $V$. Then sesquilinearity of the inner product yields

$$\langle \varphi \circ T_A^n, \psi\rangle = \sum_{k,\ell\in\mathbb{Z}^n} a_k\overline{b_\ell}\langle \chi_k \circ T_A^n, \chi_\ell\rangle.$$

Now suppose that for all $k, \ell \in \mathbb{Z}^n$ the convergence in $(\star)$ is satisfied $f = \chi_k$ and $g = \chi_\ell$, then finiteness of the above sum implies that

$$\begin{aligned}\langle \varphi \circ T_A^n, \psi\rangle &= \sum_{k,\ell\in\mathbb{Z}^n} a_k\overline{b_\ell}\langle \chi_k \circ T_A^n, \chi_\ell\rangle \\ &\overset{n\to\infty}{\longrightarrow} \sum_{k,\ell\in\mathbb{Z}^n} a_k\overline{b_\ell} \int_{\mathbb{T}^n} \chi_k \mathrm{d}m \int_{\mathbb{T}^n} \overline{\chi_\ell}\mathrm{d}m \\ &= \int_{\mathbb{T}^n} \varphi \mathrm{d}m \int_{\mathbb{T}^n} \overline{\psi}\mathrm{d}m.\end{aligned}$$

Hence, it is enough to prove this for $f = \chi_k$ and $g = \chi_\ell$ with $k, \ell \in \mathbb{Z}^n$ arbitrary.
As $(\star)$ readily follows when $k$ or $\ell$ equals 0, it is enough to show $(\star)$ for $f(x) = \chi_k(x)$ and $g = \chi_\ell(x)$ for $k, \ell \in \mathbb{Z}^n \setminus \{0\}$. Note that in this case the right-hand side of $(\star)$ is 0.

Note that

$$\chi_k \circ T_A^n(x) = \exp\left(2\pi i \langle k, A^n x\rangle\right) = \exp\left(2\pi i \langle {}^{\mathrm{t}}A^n k, x\rangle\right) = \chi_{{}^{\mathrm{t}}A^n k}(x).$$

This, together with the orthogonality relation between characters, implies that for any $k, \ell \in \mathbb{Z}^n$ with $k, \ell \neq 0$, we have

$$\langle \chi_k \circ T_A^n, \chi_\ell \rangle = \langle \chi_{{}^{\mathrm{t}}A^n k}, \chi_\ell \rangle = \begin{cases} 0, & \text{if } {}^{\mathrm{t}}A^n k \neq \ell, \\ 1, & \text{if } {}^{\mathrm{t}}A^n k = \ell. \end{cases}$$

Hence, it suffices to show that ${}^{\mathrm{t}}A^n k \neq \ell$ for all but finitely many $n \in \mathbb{N}$.
Assume for sake of contradiction that ${}^{\mathrm{t}}A^n k = \ell$ for infinitely many $n$. In particular, there exist $n_1 < n_2$ with this property, so

$${}^{\mathrm{t}}A^{n_2 - n_1} \ell = \ell.$$

This means that ${}^{\mathrm{t}}A^{n_2-n_1}$ has 1 as an eigenvalue, so ${}^{\mathrm{t}}A$ (and therefore also $A$) has an $(n_2 - n_1)$-th root of unity as one of its eigenvalues, in contradiction to (1).
Next we show that (2) implies (1). Recall that, as defined in class, $T_A$ is ergodic if the $T_A$-invariant functions in $\mathrm{L}^2(\mathbb{T}^n)$, i.e., those satisfying $f \circ T_A = f$, are exactly the (almost surely) constant functions. We will prove the contrapositive, so suppose that $A$ has an eigenvalue which is an $n$-th root of unity for some $n \in \mathbb{N} \cup \{0\}$. Therefore, $A^n$ has 1 as an eigenvalue and the same holds for ${}^{\mathrm{t}}A$. Since ${}^{\mathrm{t}}A$ is a rational matrix we can find a rational vector $0 \neq v \in \mathbb{Q}^n$ which is fixed by ${}^{\mathrm{t}}A^n$. Multiplying with the common denominator, we can assume that $0 \neq v \in \mathbb{Z}^n$. Consider now the function

$$f(x) = \sum_{m=0}^{n-1} \chi_{{}^{\mathrm{t}}A^m v}(x),$$

which is continuous and non-constant as $v \neq 0$. By the relation $\chi_k \circ T_A(x) = \chi_{{}^{\mathrm{t}}Ak}(x)$ and since ${}^{\mathrm{t}}A^n v = v$, we have

$$f \circ T_A = \sum_{m=0}^{n-1} \chi_{{}^{\mathrm{t}}A^{m+1} v} = \sum_{m=1}^{n-1} \chi_{{}^{\mathrm{t}}A^m v} + \underbrace{\chi_{{}^{\mathrm{t}}A^n v}}_{=\chi_v} = f,$$

so that $f$ is $T_A$-invariant non-constant. This shows that $T_A$ is not an ergodic transformation.

# Problem 2

This problem lists three statements. Prove exactly two of them.

**2.1** Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving system. Suppose that $T \times T$ is weak mixing. Deduce that $T$ is weak mixing.

**2.2** Let $X$ be a compact metric space and $T\colon X \to X$ a homeomorphism. Show that there exists $E \subseteq X$ closed such that $TE = E$ and such that the dynamical system $(E, T|_E)$ is minimal.

**2.3** Let

$$\begin{aligned} T_2\colon \mathbb{T} &\longrightarrow \mathbb{T}, \\ t &\longmapsto 2t. \end{aligned}$$

Show that the Lebesgue measure on $\mathbb{T}$ is $T_2$-invariant and ergodic.

**Solution:**

We prove the first two statements.

1. Suppose that $T \times T$ is weak mixing. Using the definition of weak mixing given in class, we need to show that

$$\forall A, B \in \mathcal{B} \quad \frac{1}{N} \sum_{n=0}^{N-1} \left|\mu(T^{-n}A \cap B) - \mu(A)\mu(B)\right| \stackrel{N\to\infty}{\longrightarrow} 0.$$

Let $A, B \in \mathcal{B}$ arbitrary and define $A' = A \times X$ and $B' = B \times X$. Note that $A', B' \in \mathcal{B} \otimes \mathcal{B}$ and

$$\mu(A)\mu(B) = (\mu \otimes \mu)(A')(\mu \otimes \mu)(B')$$

by definition of the product measure.

For any $n \in \mathbb{N} \cup \{0\}$ we have that

$$\begin{aligned} (T \times T)^{-n}(A') &= \{(x, y) \in X \times X \colon (T \times T)^n(x, y) \in A'\} \\ &= \{(x, y) \in X \times X \colon (T^n x, T^n y) \in A \times X\} \\ &= T^{-n}A \times X. \end{aligned}$$

In particular,

$$(T \times T)^{-n}(A') \cap B' = \Big(T^{-n}A \times X\Big) \cap (B \times X) = (T^{-n}A \cap B) \times X$$

and, hence,

$$\begin{aligned} \mu(T^{-n}A \cap B) = \mu(T^{-n}A \cap B)\mu(X) &= (\mu \otimes \mu)\Big((T^{-n}A \cap B) \times X\Big) \\ &= (\mu \otimes \mu)\Big((T \times T)^{-n}(A') \cap B'\Big). \end{aligned}$$

In particular, we have that

$$\begin{aligned}&\frac{1}{N}\sum_{n=0}^{N-1}\left|\mu(T^{-n}A\cap B)-\mu(A)\mu(B)\right|\\&\qquad=\frac{1}{N}\sum_{n=0}^{N-1}\left|(\mu\otimes\mu)\Big((T\times T)^{-n}A'\cap B'\Big)-(\mu\otimes\mu)(A')(\mu\otimes\mu)(B')\right|\overset{N\to\infty}{\longrightarrow}0.\end{aligned}$$

2. Let

$$\mathcal{E}=\{E\subseteq X\colon E\neq\emptyset,\, TE=E\ \text{ and }\overline{E}=E\}.$$

Note that $\mathcal{E}$ is non-empty since $X$ is closed and $T$ is surjective by assumption. We equip $\mathcal{E}$ with the partial order defined by inclusion, i.e.,

$$\forall E,E'\in\mathcal{E}\quad E\leq E'\text{ if }E'\subseteq E.$$

Let $\mathcal{Z}\subseteq\mathcal{E}$ be a chain. We claim that $\mathcal{Z}$ has an upper bound in $\mathcal{E}$. Indeed, let

$$\bigcap\mathcal{Z}=\bigcap_{E\in\mathcal{Z}}E.$$

Then $\bigcap\mathcal{Z}$ is an intersection of closed sets. For any finite $\mathcal{F}\subseteq\mathcal{Z}$ we know that $\bigcap_{E\in\mathcal{F}}E\neq\emptyset$, since $\mathcal{F}$ has a maximum, i.e., $\mathcal{Z}$ has the finite intersection property. Since $X$ is compact, it follows that $\bigcap\mathcal{Z}\neq\emptyset$. Indeed, assuming otherwise, we know that

$$X=\bigcup_{E\in\mathcal{Z}}(X\setminus E),$$

i.e., the collection $\{X\setminus E\colon E\in\mathcal{Z}\}$ is an open cover of $X$ and, hence, by compactness of $X$ there exists a finite set $\mathcal{F}\subseteq\mathcal{Z}$ such that

$$X=\bigcup_{E\in\mathcal{F}}(X\setminus E),$$

which implies that $\mathcal{F}\neq\emptyset$ and

$$\emptyset=X\setminus\left(\bigcup_{E\in\mathcal{F}}(X\setminus E)\right)=\bigcap_{E\in\mathcal{F}}E,$$

which is absurd.

In particular, $\bigcap\mathcal{Z}$ is a non-empty closed subset of $X$. Since $T=(T^{-1})^{-1}$ and $\mathcal{Z}\subseteq\mathcal{E}$, it follows that

$$T\bigcap\mathcal{Z}=T\left(\bigcap_{E\in\mathcal{Z}}E\right)=\bigcap_{E\in\mathcal{Z}}TE=\bigcap_{E\in\mathcal{Z}}E=\bigcap\mathcal{Z},$$

i.e., $\bigcap\mathcal{Z}\in\mathcal{E}$. This concludes the proof that $\mathcal{Z}$ has an upper bound in $\mathcal{E}$.

Using Zorn's lemma, $\mathcal{E}$ has a maximal element. We will show that any such maximal element is minimal as a topological dynamical system. So let $E\in\mathcal{E}$ maximal. Recall that $(E,T|_E)$ is minimal if every $x\in E$ has dense two-sided orbit in $E$. So let $x\in E$ arbitrary. Let $E'=\overline{\mathcal{O}(x)}$

be the closure of the two-sided orbit of $x$. Then $E' \subseteq E$. If $TE' = E'$, then $E' \in \mathcal{E}$ and, hence, maximality of $E$ implies that $E' = E$ and, since $x \in E$ was arbitrary, this then implies that $(E, T|_E)$ is minimal. Clearly $\mathcal{O}(x)$ is $T$-invariant. Hence, since $T$ is a homeomorphism, we have that

$$TE' = T\left(\overline{\mathcal{O}(x)}\right) = \overline{T\mathcal{O}(x)} = \overline{\mathcal{O}(x)} = E'.$$

This concludes the proof.

D–MATH

# Answer Booklet
# Dynamical Systems and Ergodic Theory

401-2374-24G

*Last Name*

XX

*First Name*

XX

*Legi-Nr.*

**XX-000-000**

*Exam-No.*

***000***

**Exam Duration:** 2 hours.

**Allowed aids:** None.

**Please note the following:**

- Turn off any **mobile devices and smartwatches** and keep them stored out of reach. **You may not carry any smart devices on you during the exam.**
- Place your Student ID (Legi) visibly on the table.
- Only non-erasable pens are allowed. The ink must not be red or green. Do not use whiteout, instead just cross out the relevant parts.
- Please note the **additional information** and the **list of results** on the next pages.

## Good Luck!

*Please do not fill the table!*

| | 1 | 2 | 3 | 4 | Summe |
|---|---|---|---|---|---|
| Punkte | | | | | |
| Kontrolle | | | | | |
| Maximal | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $\sum_{i=1}^{4} n_i$ |

## For answering questions:

- Questions should be **answered in this booklet**.
  - Use the pages designated for a given question.
  - Intermediate steps will be marked!
  - In case you need additional space you will be provided with additional paper. Please include a clear indication in case you answer a question somewhere other than on the designated pages of the answer booklet. Remember to mark each additional page you attach with your anonymized code from the front page.
  - You are allowed to use any results proven in class or on the problem sheets. If you do, explicitly state the result and explain clearly, how you use it in the context.
  - If you are asked to prove a result, referring to its proof in class or on the problem sheets does not constitute a solution.
  - When solving a problem, you can use without proof any results stated in the preceding subproblems or in any other problems from this exam.

## At the end of the exam

- Sort all additional pages by question.
- Wait until all exams have been collected and follow the instructions.

## Please do not remove the staple from the booklet.

**Reference results:** The following is a list of results and definitions that might come in handy. Feel free to use any of them (without proof) in the course of the exam. If you do, cite them clearly.

- Let $X$ be a compact metric space. There is a one-to-one correspondence between Borel probability mesures on $X$ and positive linear functionals $\Lambda \in \mathrm{C}(X)$ satisfying $\Lambda(1) = 1$. The correspondence is given by the map which maps a Borel probability measure $\mu$ on $X$ to the functional
$$\begin{aligned}\Lambda_\mu\colon \mathrm{C}(X) &\longrightarrow \mathbb{C},\\ f &\longmapsto \int_X f\mathrm{d}\mu.\end{aligned}$$
- Let $X$ be a compact metric space and let $\mathcal{M}_1(X)$ denote the set of Borel probability measures on $X$. Then $\mathcal{M}_1(X)$ is compact when equipped with the weak-$*$ topology, which is the minimal topology on $\mathcal{M}_1(X)$ such that for all $f \in \mathrm{C}(X)$ the map
$$\begin{aligned}\mathrm{ev}_f\colon \mathcal{M}_1(X) &\longrightarrow \mathbb{C},\\ \mu &\longmapsto \mathrm{ev}_f(\mu) = \int_X f\mathrm{d}\mu\end{aligned}$$
is continuous.
- Let $X$ be a compact metric space and $\mathcal{E} \subseteq 2^X$ a collection of closed subsets of $X$. Then
$$\forall E \in \mathcal{E} \quad \mathring{E} = \emptyset \implies \left(\bigcup_{E\in\mathcal{E}} E\right)^\circ = \emptyset.$$
- Let $X$ be a compact metric space. Then $\mathrm{C}(X)$ equipped with the metric induced by the sup-norm
$$\begin{aligned}\|\cdot\|_\infty\colon \mathrm{C}(X) &\longrightarrow [0,\infty),\\ f &\longmapsto \sup\Big\{|f(x)|\colon x \in X\Big\}\end{aligned}$$
is separable, i.e., there exists a countable dense subset $A \subseteq \mathrm{C}(X)$.
- Let $X$ be a set. A collection $\mathcal{S} \subseteq 2^X$ is a *semialgebra* if the following are true.
  - $\emptyset \in \mathcal{S}$.
  - $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$.
  - $A \in \mathcal{S} \implies X \setminus A$ is a finite union of elements in $\mathcal{S}$.

  A collection $\mathcal{A} \subseteq 2^X$ is a *algebra* if $\mathcal{A}$ is a semialgebra and the following is true.
  - $A \in \mathcal{A} \implies X \setminus A \in \mathcal{A}$.
- Let $X$ be a set and $\mathcal{S} \subseteq 2^X$ a semi-algebra. If $\mu\colon \mathcal{S} \to [0,\infty)$ is a countably additive measure, then there exists a unique countably additive measure $\tilde{\mu}\colon \sigma(\mathcal{S}) \to [0,\infty)$ such that $\tilde{\mu}|_{\mathcal{S}} = \mu$.
- Let $(X, \mathcal{B}, \mu)$ be a probability space and $\mathcal{A} \subseteq \mathcal{B}$ an algebra. Then the collection of elements $B \in \mathcal{B}$ such that for every $\varepsilon > 0$ there exists $A \in \mathcal{A}$ satisfying $\mu(A\Delta B) < \varepsilon$ is a $\sigma$-algebra.

- Let $A, B, C, D$ be sets, then
  - $A\Delta(B \cup C) \subseteq (A\Delta B) \cup (A\Delta C)$.
  - $(A \setminus B)\Delta(C \setminus D) \subseteq (A\Delta C) \cup (B\Delta D)$.

- Let $I \subseteq \mathbb{Z}$ and assume that for each $i \in I$ a measurable space $(X_i, \mathcal{B}_i)$ is given. Suppose that for every finite subset $F \subseteq I$ there is a probability measure $\mu_F$ defined on

  $$(X_F, \mathcal{B}_F) = \left(\prod_{i\in F} X_i, \bigotimes \mathcal{B}_i\right).$$

  Suppose that the measures are consistent in the sense that for every finite subset $F \subseteq I$ and for every $E \subseteq F$, the projection map

  $$(X_F, \mathcal{B}_F, \mu_F) \to (X_E, \mathcal{B}_E, \mu_E)$$

  is a measure-preserving map. Then there exists a unique probability measure $\mu$ on

  $$(X, \mathcal{B}) = \left(\prod_{i\in I} X_i, \bigotimes_{i\in I} \mathcal{B}_i\right)$$

  such that for every finite $F \subseteq I$ the projection map

  $$(X, \mathcal{B}, \mu) \to (X_F, \mathcal{B}_F, \mu_F)$$

  is measure-preserving.

# Question 1

# Question 2

# Question 3

# Question 4