Introduction

Theoretical dynamics begins with **Newton** via

(i) development of calculus

(ii) formulation of laws of motion:

(2) At any instant of time, the net force on a body is equal to the body's acceleration

"The change of ~~mot~~ motion of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed."

In formulae

$$F \propto \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

(iii) law of universal gravitation:

$$F \propto \frac{m_1 m_2}{r}.$$

How does this give rise to dynamics? Suppose we have k particles moving in $\mathbb{R}^3$ under known forces. At time t each particle has a position $q_i(t) \in \mathbb{R}^3$ $(1 \le i \le k)$ and a momentum $p_i(t) \in \mathbb{R}^3$. We assume that these determine the system completely, i.e., the state of the system at time t is an element $(p,q) \in \mathbb{R}^{3k} \times \mathbb{R}^{3k} \cong \mathbb{R}^{6k}$. The system's development over time is then assumed to be governed by, e.g., the Hamilton equations using the fct. $H(p,q)$ which describes the system at state (p,q):

$$\dot{q}_i = \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

For example, in case of a single particle in one dimension we denote by H the sum of kinetic energy T and potential energy V, ~~done~~ that is

$$H(p,q) = T(p) + V(q) = \frac{p^2}{2m} + V(q)$$

explicit form depends on problem; in physics this was discussed for different situations.

As was proven in calculus, the theorem of Picard-Lindelöf shows that under some reasonable assumptions, for any initial condition $(p(0),q(0)) = (p_0,q_0)$, this system admits a unique solution on a neighbourhood of $t_0 = 0$. The study of the behaviour of such a solution as t varies, i.e., the "trajectory" of the system parametrized by $(p(t),q(t)) \in \mathbb{R}^{6k}$ by analytic methods was roughly what was done in the aftermath by Euler, Jacobi, and so on. Note that this approach is feasible only for relatively simple problems. Famously, the three-body-problem does not admit a closed solution.

Formulating a dynamical system

Suppose that H is sufficiently nice, so that the IVP (for all IV)

$$(p(0),q(0)) = (p_0,q_0),$$

$$(\dot{p},\dot{q}) = \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$$

admits a (unique) solution. We even suppose that this solution extends to the full real line, which is the case for example if H has compact fibres, i.e., for every energy level e, the set of states $H^{-1}(e)$ of energy e is compact. Then we obtain a map

$$T_t : \mathbb{R}^{6k} \longrightarrow \mathbb{R}^{6k}, \quad (p_0,q_0) \longmapsto (p(t),q(t)) \qquad \text{"phase flow"}$$

which is a map (a continuous transformation) on $H^{-1}(H(p_0,q_0))$ for any (p_0,q_0). In calculus, you have seen that flowing first for time s and then for time t is the same as flowing for time $t+s$:

$$T_{t+s}(p_0,q_0) = T_t(T_s(p_0,q_0)),$$

i.e., $T_{\bullet} : \mathbb{R} \longrightarrow C(\mathbb{R}^{6k}, \mathbb{R}^{6k})$, $t \longmapsto T_t$

is a group homomorphism, meaning $T_{t+s} = T_t \circ T_s$ for all $t,s \in \mathbb{R}$. In particular, we obtain group actions $\mathbb{Z} \curvearrowright \mathbb{R}^{6k}$ and $\mathbb{R} \curvearrowright \mathbb{R}^{6k}$, the first by iterating, e.g., T_1, i.e., the dynamical system is given by the pair $(X,T) = (H^{-1}(e), T_1)$, where $T^{\bullet}(v) = T_1(v)$ and we are interested in the behaviour of "orbits" (or curves)

$$\{T^n(v) : v \in X, n \in \mathbb{Z}\} = \{T_n(p_0,q_0) : (p_0,q_0) \in \mathbb{R}^{6k}, n \in \mathbb{Z}\}.$$

This abstraction is useful! One of the main original contributions was:

End of 19^{th} century – Poincaré ("Méthodes nouvelles de la mécanique céleste.")

Paradigm: Don't study individual orbits/curves, but study the collection of all orbits/curves by studying the geometry of the phase space.

Remark: The geometry of the phase space tends to be quite interesting. For example, energy preservation implies that any solution to

$$(p(0), q(0)) = (p_0, q_0), \quad (\dot p, \dot q) = \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$$

satisfies that $t \longmapsto H(p(t), q(t))$ is constant, i.e., the curves are contained in level sets $H^{-1}(e)$, for $e = H(p_0, q_0)$. The manifold $H^{-1}(e)$ has its own geometry.

In this contribution, Poincaré showed that the general solution to the three-body problem does not admit a closed form/expression in terms of algebraic + transcendental functions.

So far, we see that we are interested in dynamical systems. What about Ergodic theory?

Recall the setup $H(p,q)$ with

$$\dot p = -\frac{\partial H}{\partial q}, \quad \dot q = \frac{\partial H}{\partial p}.$$

Write $F(p,q) := \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$, which describes the ODE of interest. Note that

$$\operatorname{div} F := \nabla \cdot F = \frac{\partial}{\partial p} F_p + \frac{\partial}{\partial q} F_q = \frac{\partial}{\partial p}\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p}\right) = 0$$

as you have shown in calculus.

Liouville's theorem (special case)

Let $(\dot p, \dot q) = F(p,q)$ with $\operatorname{div} F = 0$, then

$$T_t : \mathbb{R}^{6k} \longrightarrow \mathbb{R}^{6k}$$

preserves the volume (for arbitrary t), i.e., for all bounded (measurable) $B \subseteq \mathbb{R}^{6k}$ we have that

$$\operatorname{vol}(T_t B) = \operatorname{vol}(B).$$

Sketch of proof: For small t, we have $T_t x = x + F(x) t + O(t^2)$. Hence

$$D_x T_t = id + \frac{\partial F}{\partial x} \cdot t + O(t^2).$$

We compute

$$\operatorname{vol}(T_t B) = \int_{T_t B} 1 \, dx = \int_B |\det(D_x T_t)| \, dx = \int_B \left|\det\left(id + \frac{\partial F}{\partial x} t + O(t^2)\right)\right|$$

As shown in calculus, we have

$$\det\left(id + \frac{\partial F}{\partial x} t + O(t^2)\right) = 1 + \operatorname{tr}\left(\frac{\partial F}{\partial x}\right) t + O(t^2)$$
$$= 1 + \operatorname{div} F \cdot t + O(t^2),$$

hence

$$\operatorname{vol}(T_t B) = \int_B 1 + t \operatorname{div}(F) + O(t^2) \, dx,$$

thus $\frac{d}{dt} \operatorname{vol}(T_t B) = 0$. $\boxed{?}$

Rmk: If $H^{-1}(e)$ ~~Th~~ is compact, T_t preserves the finite measure on $H^{-1}(e)$.

This motivates the study of dynamical systems which preserve a (finite) measure, so-called (probability) measure preserving systems.

Now enters

Boltzmann's hypothesis in statistical mechanics. For some reason, we face a bounded system in 6k-dimensional phase space whose evolution is governed by Hamilton's equations and a subdivision of that phase space into cells Δ of equal volume, related to the highest precision with which we presumably are able to measure positions and momenta or times and energies: $H^{-1}(e) = \bigsqcup_{i=1}^{N} \Delta_i$

Then the evolution can be viewed as a permutation of the cells of equal energy. In this context, Boltzmann's ergodic hypothesis becomes that every cell of a given energy visits all the others, or

"The action of the evolution transformation T, w.r.t. a given unit time as a cell permutation of the phase space cells on the surface of constant energy, is a one cycle permutation of the N phase space cells with the given energy e:

$$T\Delta_i = \Delta_{i+1} \quad (\text{where } \Delta_{N+1} := \Delta_1);$$

cf. Gallavotti – Statistical Mechanics."

~~Put differently, in our context we want~~

If not true this strictly, it should at least be true for the purpose of computing time averages, which in our notation becomes

$$\frac{1}{R}\int_0^R \mathbb{1}_B(T_t(p_0,q_0))\,dt \xrightarrow{R\to\infty} \mathrm{vol}(B)$$

for "most" $(p_0,q_0)\in H^{-1}(e)$. Ergodicity arose as the property to impose to guarantee such convergence. The word "ergodic" was coined from

ἔργον = "ergon" = work,

ὁδός = "hodos" = path.

Using the general notion of dynamical systems one can in some instance, for example, prove the existence of periodic orbits, e.g., like planetary orbits. This way to obtaining this comforting insight is clearly superior to checking all possible initial conditions one-by-one, in particular if the general solution doesn't have a closed form (e.g., 3-body problem).

Recap: Following the preceding discussion, we are left to study

X = a space

G = a group $\circlearrowright X$.

Hence, in some sense, a dynamical system is just a pair (X,G) with an implicit understanding that we are most interested in the orbits of the action, i.e., the maps

$$\alpha_x : G \longrightarrow X,$$
$$g \longmapsto T_g x.$$

In this course we will be most interested in ~~the~~ actions of $\mathbb{Z}$ (or $\mathbb{N}$) which arise by iterating a single map $T: X \longrightarrow X$ (required to be invertible for $\mathbb{Z}$-actions). Such maps arise for example as in the Boltzmann hypothesis by choice of a unit time, i.e.,

$$T = T_1.$$

In order to develop useful theory, we have to introduce additional structure. We look at two types of dynamical systems

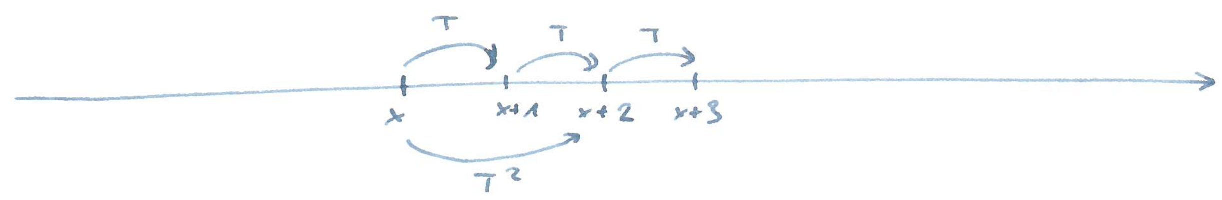
top. dynamics ------> 1) X is a topological space and T is a homeomorphism
(cpt. metric)

ergodic theory ------> 2) $(X,\mathcal{B},\mu)$ is a probability space and T is measurable and measure preserving, i.e., $\forall B\in\mathcal{B}\quad \mu(T^{-1}B)=\mu(B)\quad (T_*\mu=\mu)$

A list of (~~topological~~) examples

Example 1:

Let $X=\mathbb{R}$, $T:\mathbb{R}\longrightarrow\mathbb{R}$
$x \longmapsto x+1$



Properties: – T is a homeo
– all orbits look the same
– every orbit diverges

Hence in the examples to follow, we require some boundedness in the hope to guarantee some more interesting dynamics.

Notation: Given a pair (X,T) consisting of a space X and a map $T: X\longrightarrow X$, we denote for $x\in X$

$$\mathcal{O}(x) = \{T^n x : n\in\mathbb{Z}\} \qquad (\text{if } T \text{ invertible}),$$
$$\mathcal{O}^+(x) = \{T^n x : n\in\mathbb{N}\}.$$

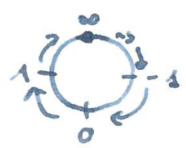
Remark: In the discussion before, involving Hamiltonians, X was the energy surface $H^{-1}(e)$ within the phase space $\mathbb{R}^{6k}$ and $T=T_{t_0}$ for some unit time t_0. In particular, X was a topological space and T was a homeomorphism. In this setup, Poincaré showed "stability in the sense of Poisson", i.e., $X=H^{-1}(e)$ was compact and

$$x\in\overline{\mathcal{O}^+(x)} \qquad (\text{"recurrence"})$$

for almost every $x\in H^{-1}(e)$.

Example 2a: (Compactification of the previous example)

We compactify the space in the previous example and extend the map by setting $X=\mathbb{R}\cup\{\infty\}$, $T: X\longrightarrow X$, $T(\infty):=\infty$.



As seen in topology, this is the Alexandrov/one-point compactification of $\mathbb{R}$ and the extension T is the unique continuous extension.

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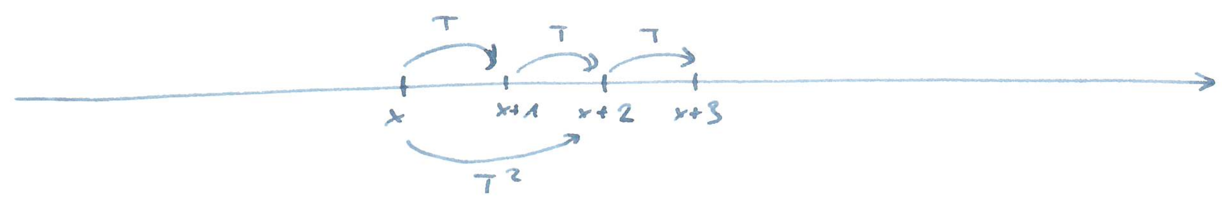

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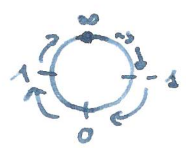

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Now we observe some different behaviour:

- ∞ is a fixed point.
- $\forall x \in X \ \lim_{n\to\infty} T^n x = \infty$.

Definition (ω-limit sets)

Let X be a (compact metric) topological space and suppose $T: X \longrightarrow X$ is continuous.

The (forward) ω-limit set is

$$\omega^+(x) = \{ y \in X : \exists (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}} \text{ s.t. } n_k \xrightarrow{k\to\infty} \infty \text{ and } \lim_{k\to\infty} T^{n_k} x = y \}.$$

If T is a homeomorphism, we let

$$\omega^{\pm}(x) = \{ y \in X : \exists (n_k)_{k\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} \text{ s.t. } |n_k| \xrightarrow{k\to\infty} \infty \text{ and } \lim_{k\to\infty} T^{n_k} x = y \}.$$

$$\omega^-(x) = \{ y \in X : \text{" " " " s.t.} -n_k \to \infty \text{ " " " "} \}$$

sometimes α-limit set

In the previous example

$\forall x \in X \quad \omega^+(x) = \{\infty\} = \omega^{\pm}(x)$

$\forall x \in X - \{\infty\} \quad \omega^-(x) = \emptyset$

$\omega^-(\infty) = \{\infty\}$.

Also note that ∞ is a special case of the following

Definition (periodic points)

Given $T: X \longrightarrow X$ (for X a set), a point $x \in X$ is periodic if $\exists n \in \mathbb{N}$ s.t. $T^n x = x$.

Remark: T has periodic points if and only if some power of T has fixed points.

- Sometimes, it'll be useful to denote

$$\mathrm{Fix}(T) = \{ x \in X : Tx = x \},$$

$$\mathrm{Per}(T) = \{ x \in X : \exists n \in \mathbb{N} \text{ s.t. } T^n x = x \}.$$

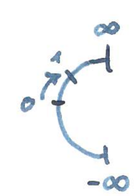
Example 2b: (Another compactification)

Let $X = \mathbb{R} \cup \{\infty, -\infty\}$ ("two-point" compactification). The topology can be obtained as the topology induced by the identification $X \cong [0,1]$ given by

$$t \longmapsto \frac{e^t}{1+e^t} \quad \text{(inverse of exp of logit map)}.$$

We define $T: X \to X$ by

$$T(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{R}, \\ \infty & \text{if } x = \infty, \\ -\infty & \text{if } x = -\infty. \end{cases}$$



$\forall x \in \mathbb{R}$: $\{\infty\} = \omega^+(x)$

$\{-\infty\} = \omega^-(x)$

$\{\infty, -\infty\} = \omega^{\pm}(x)$

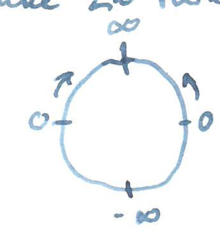
$\omega^+(\infty) = \omega^-(\infty) = \omega^{\pm}(\infty) = \{\infty\}$

Similarly for $-\infty$.

Then $\mathrm{Fix}(T) = \{+\infty, -\infty\}$. Exercise: Determine all the ω-limit sets.

Example 2c: (North-South dynamics)

We take 2b twice and glue the endpoints



In all these examples, periodic orbits and ω-limits existed and consisted of fixed points. The fixed pts in b & c are attractive ($+\infty$) and/or repulsive ($-\infty$) (every pt. in a neighbourhood of x (except x) stays or eventually leaves that nghd.). The fixed points in a are neither.

Example 3: (Circle rotation)

Let $X = \mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$, i.e., X is the circle obtained by identifying any two real numbers that differ by an integer. This is a compact metric space and, in fact, a topological group, so we can look at the map obtained by multiplication by a given group element. More explicitly, let $\alpha \in \mathbb{T}$, then we denote

$$R_\alpha : \mathbb{T} \longrightarrow \mathbb{T},$$
$$x \ (\mathrm{mod}\ 1) \longmapsto x + \alpha \ (\mathrm{mod}\ 1).$$

In the circle interpretation, this corresponds to

$$z \in \mathbb{S}^1 \longmapsto e^{2\pi i \alpha} \cdot z \in \mathbb{S}^1.$$

In this case we observe two fundamentally different patterns depending on α.

- if α is rational, i.e., $\alpha \in \mathbb{Q}/\mathbb{Z}$, suppose $\alpha = \frac{p}{q} \bmod 1$ with p, q coprime, then

$$\forall x \in \mathbb{T} \quad R_\alpha^q(x) = x + q\alpha \bmod 1 = x + p \bmod 1 = x \bmod 1.$$

In other words, every $x \in \mathbb{T}$ is periodic with period (exactly) q.

- if $\alpha \notin \mathbb{Q}/\mathbb{Z}$, then no point is periodic. To see this, let $q \in \mathbb{N}$ and suppose that $x \in \mathbb{T}$ satisfies

$$R_\alpha^q(x) = x.$$

This means

$$x + \alpha q \equiv x \bmod 1 \iff \alpha q \equiv 0 \bmod 1 \iff \alpha q \in \mathbb{Z} \iff \alpha \in \mathbb{Q}.$$

In fact:

If $\alpha \notin \mathbb{Q}/\mathbb{Z}$, then $\forall x \in \mathbb{T} \ \overline{\mathcal{O}^+(x)} = \mathbb{T}$.

End of lecture 1