Example 4: (Times p / xp)

Again, $X = \mathbb{T}$. Let $p \in \mathbb{Z} - \{0\}$ (usually $p \in \mathbb{N}$). Define $T : X \longrightarrow X,$
$$x \longmapsto px \bmod 1.$$

This is well-defined ($\mathbb{T}$ is an abelian group): T is a continuous surjective group homomorphism.

- If $p = 1$, then $T = id$ and hence every pt. is a fixed point.
- If $p = -1$, then $T^2 = id$ and hence every pt. has periodic orbit.
- If $p \notin \{\pm 1\}$, let $x \in \mathbb{T}$ and suppose $k \in \mathbb{N}$ satisfies
$$T^k x = x \iff p^k x = x + \mathbb{Z}, \text{ i.e., } (p^k - 1)x \in \mathbb{Z}.$$

In particular, T has many periodic points of varying period k for every $k \in \mathbb{N}$.

Exercise: Suppose $p \in \mathbb{N} - \{1\}$.

a) Find x s.t. $\omega^+(x) = \mathbb{T}$.

b) Find x s.t. $\omega^+(x)$ is uncountable but not equal to $\mathbb{T}$.

c) Find x s.t. $\omega^+(x)$ is countably infinite.

d) Find x s.t. $\omega^+(x)$ is finite but x is not periodic.

Hint: Given $x \in \mathbb{T} - \{0 + \mathbb{Z}\} \cong (0,1)$, consider its base-$p$ expansion
$$x = \sum_{k=1}^{\infty} \frac{a_k}{p^k} \qquad (a_k \in \{0, \dots, p-1\})$$
and note that
$$Tx = \sum_{k=1}^{\infty} \frac{a_{k+1}}{p^k}.$$

Example 5 (Bernoulli shift)

Fix an alphabet $A = \{0, \dots, k-1\}$ and look at

- $X = A^{\mathbb{N}}$ and $T((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$ (one-sided Bernoulli shift),
- $X = A^{\mathbb{Z}}$ and $T((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{Z}}$ (two-sided Bernoulli shift).
$$(\dots, x_{-2}, x_{-1}, \overset{\downarrow}{x_0}, x_1, x_2, \dots) \mapsto (\dots, x_{-1}, x_0, \overset{\downarrow}{x_1}, x_2, x_3, \dots)$$

Remark: In both cases, X is a compact metric space with metric defined as follows.

Given $(x_n) \in X, (y_n) \in X$, let
$$N((x_n),(y_n)) := \min\{N : x_N \neq y_N \text{ or } x_{-N} \neq y_{-N}\}$$
and define $d((x_n),(y_n)) := \begin{cases} \left(\frac{1}{2}\right)^{N((x_n),(y_n))} & \text{if } (x_n) \neq (y_n), \\ 0 & \text{else.} \end{cases}$

Then T is a continuous map and, in the case of the two-sided shift*, even a homeomorphism.

We leave the proof as an exercise and return to it when we have to look at countable products of general compact metric spaces. In the most general version, this will be discussed in the topology class.

Example 6 (Hyperbolic toral automorphism)

Let $X = \mathbb{T}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{T}\}$. Let $T_A : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$
$$(x_1, x_2) \longmapsto (x_2, x_1 + x_2) = (x_1, x_2) \underbrace{\overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{=:A}}_{\in GL_2(\mathbb{Z})}.$$

Then T_A is invertible with inverse
$$T_{A^{-1}}(x_1, x_2) = (x_2 - x_1, x_1) = (x_1, x_2)\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = (x_1, x_2)A^{-1}.$$

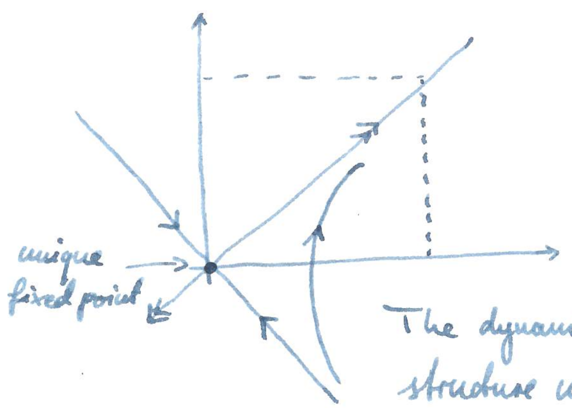
This is a homeomorphism of $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$.

Exercise:

- T_A has a dense set of periodic points.
- T_A has "even more" non-periodic points.

Remark: The matrix A has two real eigenvalues λ_1, λ_2 with
$$0 < |\lambda_1| < 1 < |\lambda_2| \qquad \left(\lambda_i = \frac{1 \pm \sqrt{5}}{2}\right).$$

The eigenspaces define so-called stable/unstable manifolds at every $x \in \mathbb{T}^2$:



The dynamical system is hyperbolic because of the orbit structure near the fixed point; there the dynamics is "chaotic".

Example 7: (Gauss map)

Let $X = (0,1) - \mathbb{Q}$. Define $Tx = \left\{\frac{1}{x}\right\} := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$. This map gives the dynamic interpretation of the continued fraction expansion: If $x \in (0,1) - \mathbb{Q}$ $\exists! (a_1, a_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ s.t.
$$x = \lim_{n \to \infty} \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots \frac{1}{a_n}}}}} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

and the relation is given by

$$Tx = \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}$$

or, alternatively $T(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$.

T is not invertible and defined on a weird space. Here, we will see that we are dealing with a pmps ($\mathbb{Q} \cap (0,1)$ is Lebesgue null in $(0,1)$).

Example 8 (Rationality detection)

Here is a system that detects whether a real number is rational. Look at

$$T : [0,1] \longrightarrow [0,1],$$
$$x \longmapsto Tx = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 1, \\ nx - 1 & \text{if } x \in \left[\frac{1}{n}, \frac{1}{n-1}\right). \end{cases}$$

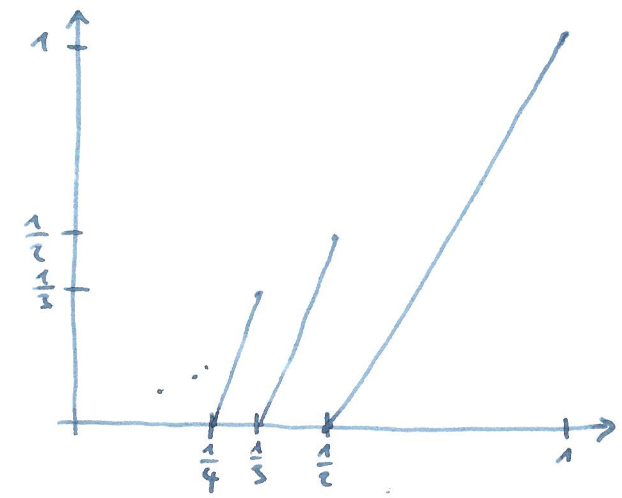
That is, T is defined piece-wise on

$$[0,1] = \{0,1\} \sqcup \bigsqcup_{\substack{n \in \mathbb{N} \\ n > 1}} \left[\frac{1}{n}, \frac{1}{n-1}\right)$$

Note that $\forall x \in \left[\frac{1}{n}, \frac{1}{n-1}\right)$

$$0 \le nx - 1 < n \cdot \frac{1}{n-1} - \frac{n-1}{n-1} = \frac{1}{n-1},$$

and $\lim\limits_{x \uparrow \frac{1}{n-1}} nx - 1 = \frac{1}{n-1}$. Hence the graph of T is



Exercise: $x \in \mathbb{Q} \cap [0,1] \iff \exists n \in \mathbb{N}$ s.t. $T^n x = 0$.

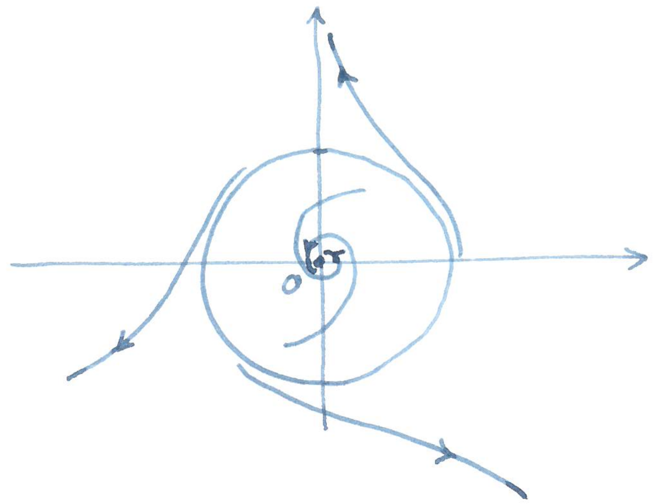
Example 9 (Quadratic family) (Polynomial, non-invertible, chaotic in parts of the space)

Let $c \in \mathbb{C}$ fixed and define

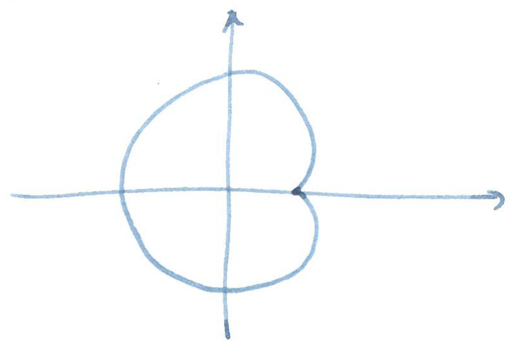
$$T : \mathbb{C} \longrightarrow \mathbb{C},$$
$$z \longmapsto z^2 + c.$$

This is a well-studied example both in mathematics and computer science, I was told. This leads to the notion of Julia and Fatou sets, which are closely connected to the famous Mandelbrot set.

For $c = 0$, $z = 0$ is a fixed point and attractive: all pts. in a neighbourhood of 0 are attracted towards 0.

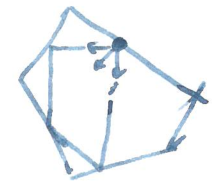


More generally, $z_0^2 + c = z_0 \iff z_0 = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ with one solution close to 0 if $c \approx 0$. The fixed point is attractive if $|T'_{z_0}| < 1 \iff |2z_0| < 1 \iff |1 - \sqrt{1 - 4c}| < 1$, which defines a cardioid, the big one in the Mandelbrot set.



Example 10: (Billiards)

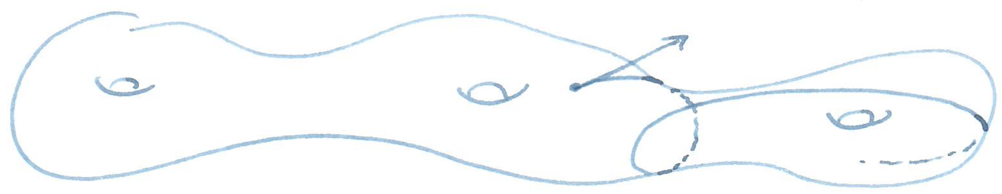
Given a "table" one can define a billiard map either by describing the full path, i.e., the "flow", or by describing the next collision with the boundary (point and direction)



Big open question: If the table is triangular, is there a periodic path?

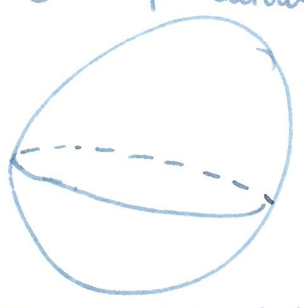
Example 11 (Geodesic flow)

Given a (Riemann or even hyperbolic) surface, one can make sense of a geodesic flow that follows a given tangent vector along a "straight line":



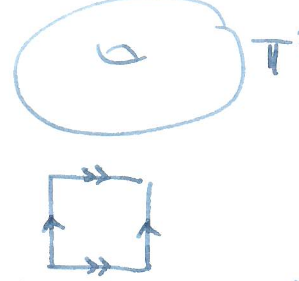
Depending on the structure of the surface, the dynamics changes dramatically:

const. pos. curvature



Every geodesic orbit is periodic ("great circle")

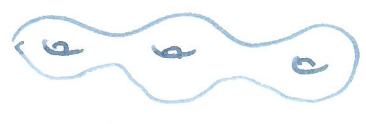
zero curvature



Geodesic orbits are either periodic or dense

"easy dynamics"

const. negative curvature



"chaotic dynamics", there are periodic orbits, dense orbits, and many in between.

I – Topological dynamics

In this chapter, X is a compact metric space and $T: X \longrightarrow X$ is continuous, so the pair (X, T) is our dynamical system.

I.1 – Generalities

Definition

- T is (topologically) forward transitive if
$$\exists x_0 \in X \text{ s.t. } \overline{\mathcal{O}^+(x_0)} = X.$$
- If T is a homeomorphism, then T is (topologically) transitive if
$$\exists x_0 \in X \text{ s.t. } \overline{\mathcal{O}(x_0)} = X.$$

Exercise: Suppose that T is a homeomorphism. Show that $\overline{\mathcal{O}(x_0)}$ is T-invariant, i.e.,
$$\forall x_0 \in X \quad T\left(\overline{\mathcal{O}(x_0)}\right) = \overline{\mathcal{O}(x_0)}.$$

Example: $T_p: \mathbb{T} \longrightarrow \mathbb{T}$ for $p \in \mathbb{N}$ is forward transitive. To this end let $\Lambda = \bigcup_{n \in \mathbb{N}} \{0, \dots, p-1\}^n$ be the collection of all finite words in the alphabet $\{0, \dots, p-1\}$. This is a countable union of finite sets, hence countable and we can enumerate it, e.g.,
$$\underset{a_1}{\underset{\shortparallel}{\omega_1 = (0)}}, \dots, \omega_p = \underset{a_p}{\underset{\shortparallel}{(p-1)}}, \omega_{p+1} = (\underset{a_{p+1}}{\underset{\shortparallel}{0}}, \underset{a_{p+2}}{\underset{\shortparallel}{0}}), \omega_{p+2} = (\underset{a_{p+3}}{\underset{\shortparallel}{0}}, \underset{a_{p+4}}{\underset{\shortparallel}{1}}), \dots$$
and so on. Define $x_0 \in \mathbb{T}$ by
$$x_0 = \sum_{n \geq 1} \frac{a_n}{p^n} = \frac{1}{p^2} + \frac{2}{p^3} + \dots + \frac{p-1}{p^p} + \frac{1}{p^{p+4}} + \dots$$

Claim:
Let $l \in \mathbb{N}$, $0 < k < p^l$, then $\exists n \in \mathbb{N}$ s.t. $T_p^n(x_0) \in \left[\frac{k}{p^l}, \frac{k+1}{p^l}\right)$.

Proof: Expand k in base p: $k = \sum_{i=0}^{l-1} b_i 2^i$, then we choose $j \in \mathbb{N}$ s.t.
$$\omega_j = (b_{l-1}, \dots, b_0)$$
and note that for some $n \in \mathbb{N} \cup \{0\}$ we have $\omega_j = (a_{n+1}, \dots, a_{n+l})$ and, hence,
$$T_p^n(x_0) = \frac{b_{l-1}}{p} + \dots + \frac{b_0}{p^l} + r,$$
where $|r| \overset{\text{some } a_i \neq p-1}{<} \sum_{n > l} \frac{p-1}{p^n} = \frac{1}{p^l}$. Hence $\frac{k}{p^l} \leq T_p^n(x_0) < \frac{k+1}{p^l}$.

End of lecture 2 ☒