Example: $-R_\alpha : \mathbb{T} \longrightarrow \mathbb{T}$. is <u>not</u> top. transitive for $\alpha \in \mathbb{Q}/\mathbb{Z}$,

. is top. transitive for $\alpha \notin \mathbb{Q}/\mathbb{Z}$.

~~We need the following finite intersection property for compact~~

Proposition

Let $T : X \longrightarrow X$ a homeomorphism. TFAE:

(i) T is transitive

(ii) If $U \subseteq X$ is open with $TU = U$, then either U is dense or $U = \emptyset$.

(iii) If $U, V \subseteq X$ are open non-empty, then $\exists n \in \mathbb{Z}$ s.t

$$T^n U \cap V \neq \emptyset.$$

(iv) The set of points with dense orbit is a dense G_δ-set.

Remark: $A \subseteq X$ is a G_δ-set if $A = \bigcap_{n\in\mathbb{N}} U_n$ for $U_n \subseteq X$ open.

Here, G = "Gebiet" and δ = "Durchschnitt".

For the proof we need the following special case of the Baire category theorem.

Lemma

Let X compact metric and let $\{A_n : n \in \mathbb{N}\}$ a sequence of closed subsets.

$$\forall n \in \mathbb{N} \;\; \mathring{A}_n = \emptyset \Longrightarrow \Big(\bigcup_{n\in\mathbb{N}} A_n\Big)^\circ = \emptyset$$

Here, $\mathring{A}_n = \{x \in A_n : \exists r > 0 \text{ s.t. } B_r(x) \subseteq A_n\}$.

Proof: Step 1: $X - A_n \neq \emptyset$.

Suppose $X - A_n = \emptyset \Rightarrow X \subseteq A_n \Rightarrow B_r(x) \subseteq A_n$ for all $r > 0$ and for all $x \in X$.

Step 2: $\forall n \in \mathbb{N}$ $\big(\bigcup_{k=1}^n A_k\big)^\circ = \emptyset$. Clear for $n = 1$. We argue by induction.

Suppose $\big(\bigcup_{k=1}^{n+1} A_k\big)^\circ \neq \emptyset \Rightarrow \exists\, x \in X\; \exists r > 0$ s.t. $B_r(x) \subseteq \bigcup_{k=1}^{n+1} A_k$. Since

A_{n+1} is closed and $\mathring{A}_{n+1} = \emptyset$, we know that $B_r(x) \not\subseteq A_{n+1}$. Hence, since

A_{n+1} is closed, there is $\underset{\delta>0}{y\in B_r(x)}$ s.t. $B_\delta(y) \subseteq B_r(x)$ and $B_\delta(y) \cap A_{n+1} = \emptyset$.

In particular $B_\delta(y) \subseteq \bigcup_{k=1}^{n} A_k$, which contradicts the induction hypothesis.

Step 3: Suppose $\big(\bigcup_{n\in\mathbb{N}} A_n\big)^\circ \neq \emptyset$. Let $x \in X, r > 0$ s.t. $B_r(x) \subseteq \bigcup_{n\in\mathbb{N}} A_n$. Given $n \in \mathbb{N}$, define

$$U_n = B_r(x) \cap \Big(\bigcup_{k=1}^n A_k\Big)^{c}.$$

Then U_n is non-empty and contained in $B_r(x)$ (by step 2). Note: $U_{n+1} \subseteq U_n$

Now we inductively construct the following sets. Choose $x_1 \in U_1$ and $r_1 > 0$ such that $\overline{B_{r_1}(x_1)} \subseteq U_1$. We denote $V_1 = B_{r_1}(x_1)$. Given V_n, choose $x_{n+1} \in X$ and $r_{n+1} > 0$ such that

$$\overline{B_{r_{n+1}}(x_{n+1})} \subseteq V_n \cap U_{n+1}$$

and denote $V_{n+1} := B_{r_{n+1}}(x_{n+1})$.

The sets $\overline{V_n}$ are compact and non-empty and $\overline{V_n} \subseteq V_{n-1}$.

Claim:

$$\emptyset \neq \bigcap_{n\in\mathbb{N}} \overline{V_n}$$

Define $O_n = X - \overline{V_n}$ and suppose for sake of contradiction that

$$X = \bigcup_{n\in\mathbb{N}} O_n.$$

Since X is compact, there is $F \subseteq \mathbb{N}$ finite such that

$$X = \bigcup_{n\in F} O_n \iff \emptyset = \bigcap_{n\in F} O_n^c = \bigcap_{n\in F} \overline{V_n} = \overline{V_{\max F}}$$

$\lightning$ $\blacksquare$

Remark: The last step didn't use countability!

We now can prove the proposition:

(i) $\Rightarrow$ (ii): Suppose $U \subseteq X$ is non-empty and $TU = U$. Using the transitivity, choose $x_0 \in X$ s.t. $\overline{\mathcal{O}(x_0)} = X$. Hence, there is $n \in \mathbb{Z}$ such that $T^n(x_0) \in U$. Since $TU = U$, we find that $T^k(x_0) \in U$ for all $k \in \mathbb{Z}$. In particular

$$X = \overline{\mathcal{O}(x_0)} \subseteq \overline{U}.$$

(ii) $\Rightarrow$ (iii): Suppose $U, V \subseteq X$ non-empty. Define $\tilde{U} := \bigcup_{n\in\mathbb{Z}} T^n U$. Then $\tilde{U}$ is open, non-empty, and T-invariant. Hence $\tilde{U}$ is dense, thus

$$\tilde{U} \cap V \neq \emptyset \Rightarrow \exists n \in \mathbb{Z} \;\; T^n U \cap V \neq \emptyset.$$

(iii) $\Rightarrow$ (iv): We define a sequence of covers of X as follows. For every $n \in \mathbb{N}$, let

$$\{B_{\frac{1}{n}}(x) : x \in X\} =: \mathcal{F}_n,$$

then $\mathcal{F}_n$ is a covering of X. Hence $\mathcal{F}_n$ admits a finite subcover $\mathcal{E}_n$.

We let $\mathcal{E} := \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$ and note that $\mathcal{E}$ is a countable collection of sets.

Define $G = \bigcap_{U\in\mathcal{E}} \bigcup_{n\in\mathbb{Z}} T^n U$.

Note that for every $U \in \mathcal{E}$, the set
$$\bigcup_{n\in\mathbb{Z}} T^n U \subseteq X$$
is open and dense.

Claim:

G is dense

Given $U \in \mathcal{E}$, let $O_U = \bigcup_{n\in\mathbb{Z}} T^n U$ and $A_U = X - O_U$. Since O_U is open, dense, we know that $A_U^\circ = \emptyset$ for all $U \in \mathcal{E}$. Using the special case of the Baire category theorem, it follows that
$$\left(\bigcup_{U\in\mathcal{E}} A_U\right)^\circ = \emptyset,$$
i.e., $\forall x \in X \, \forall r > 0$
$$B_r(x) \cap \left(X - \bigcup_{U\in\mathcal{E}} A_U\right) \neq \emptyset.$$

But
$$X - \bigcup_{U\in\mathcal{E}} A_U = \left(\bigcup_{U\in\mathcal{E}} A_U\right)^c = \bigcap_{U\in\mathcal{E}} A_U^c = \bigcap_{U\in\mathcal{E}} O_U = G.$$

~~Claim~~ Claim:

$x \in X$ has dense orbit iff $x \in G$.

We show that $x \in X$ has dense orbit iff $\forall U \in \mathcal{E} \, \exists k \in \mathbb{Z}$ s.t. $T^k x \in U$.

"$\Rightarrow$": Suppose that $U \in \mathcal{E}$ and $T^k x \notin U$ for all $k \in \mathbb{Z}$, then certainly x has non-dense orbit.

"$\Leftarrow$": Let $y \in X$ and $r > 0$ arbitrary. There exists $0 < \varrho < \frac{r}{2}$ and $y' \in X$ s.t.

~~B~~ $y \in B_\varrho(y') \in \mathcal{E}$.

Let $k \in \mathbb{Z}$ s.t. $T^k x \in B_\varrho(y')$. Then $T^k x \in B_r(y)$, since
$$B_\varrho(y') \subseteq B_r(y).$$

Clearly
$$\{x \in X : \forall U \in \mathcal{E} \, \exists k \in \mathbb{Z} : T^k x \in U\} = \bigcap_{U\in\mathcal{E}} \bigcup_{k\in\mathbb{Z}} T^k U.$$

(iv) $\Rightarrow$ (i): This is clear.

Lemma

Suppose $E = TE$ for E closed and suppose $x \in E$ satisfies
$$\overline{\mathcal{O}(x)} = X.$$
Then $E = X$. That is, the only closed subset of X containing a topologically transitive point ~~of~~ is X itself.

Proof: This is clear: $E = TE \Rightarrow \mathcal{O}(x) \subseteq E \Rightarrow X = \overline{\mathcal{O}(x)} \subseteq \overline{E} = E$.

Definition (Minimality)

A homeomorphism $T: X \to X$ is <u>minimal</u> if
$$\forall x \in X \quad \overline{\mathcal{O}(x)} = X \qquad (\mathcal{O}^+(x) \text{ if } T \text{ not a homeo})$$

Note: Minimality $\Rightarrow$ Transitivity.

Claim: Let $\alpha \in \mathbb{T} - \mathbb{Q}/\mathbb{Z}$, then R_α is minimal (later)

Claim: Let $p \in \mathbb{Z} - \{0, 1, -1\}$. Then T_p is <u>not</u> minimal (for forward orbits).

For example, there are plenty of periodic points.

Proposition

Let X compact metric, $T: X \to X$ a homeomorphism. TFAE:

(i) T is minimal

(ii) If $E \subseteq X$ is closed and $TE = E$, then either $E = X$ or $E = \emptyset$.

(iii) If $U \subseteq X$ is open non-empty, then
$$X = \bigcup_{n\in\mathbb{Z}} T^n U$$

Proof: (i) $\Rightarrow$ (ii): Cf. previous lemma.

(ii) $\Rightarrow$ (iii): Let $U \subseteq X$ open non-empty, then $\bigcup_{n\in\mathbb{Z}} T^n U$ is dense and open. Its complement is closed and T-invariant, hence empty.

(iii) $\Rightarrow$ (i): Let $U \subseteq X$ open and $x \in X$ arbitrary. If U is non-empty, then
$$x \in \bigcup_{n\in\mathbb{Z}} T^n U \Rightarrow \exists n \in \mathbb{Z} \text{ s.t. } T^n x \in U.$$
As U was arbitrary, $\mathcal{O}(x) \subseteq X$ is dense. Since x was arbitrary, T is minimal. $\blacksquare$

Theorem

Let $T: X \to X$ a homeomorphism of a compact metric space. Then $\exists E \subseteq X$ closed non-empty such that $TE = E$ and $(E, T|_E)$ is minimal.

Rmk: $X - E$ is not compact, hence we can not continue decomposing X.

Proof: We consider
$$\mathcal{Z} = \{Z \subseteq X \mid Z \text{ non-empty closed and } TZ = Z\}.$$
Then

- $\mathcal{Z}$ is non-empty since $X \in \mathcal{Z}$.
- $\mathcal{Z}$ is partially ordered by inclusion
  - $\forall Z \in \mathcal{Z} \quad Z \subseteq Z$ (reflexivity)
  - $\forall Z_1, Z_2 \in \mathcal{Z} \quad Z_1 \subseteq Z_2 \wedge Z_2 \subseteq Z_1 \Rightarrow Z_1 = Z_2$ (antisymmetry)
  - $\forall Z_1, Z_2, Z_3 \in \mathcal{Z} \quad Z_1 \subseteq Z_2, Z_2 \subseteq Z_3 \Rightarrow Z_1 \subseteq Z_3$ (transitivity)
- Every chain has a lower bound in $\mathcal{Z}$, i.e., suppose $\mathcal{C} = \{Z_i : i \in I\} \subseteq \mathcal{Z}$ is totally ordered, i.e., $\forall i, j \in I \quad Z_i \subseteq Z_j \vee Z_j \subseteq Z_i$, then there is $Z \in \mathcal{Z}$ such that $Z \subseteq Z_i$ for all $i \in I$.

  To see this, let $\mathcal{C}$ be a chain and define
  $$Z = \bigcap_{i \in I} Z_i.$$
  Then Z is closed and $TZ = Z$. Hence we only need to show that Z is non-empty. Assume otherwise, i.e., suppose $Z = \emptyset$. Define $O_i := X - Z_i$. Then $O_i \subseteq X$ is open and
  $$\bigcup_{i \in I} O_i = \bigcup_{i \in I} Z_i^c = \left(\bigcap_{i \in I} Z_i\right)^c = X.$$
  Hence there is $F \subseteq I$ finite such that
  $$X = \bigcup_{i \in F} O_i = \left(\bigcap_{i \in F} Z_i\right)^c,$$
  i.e.,
  $$\bigcap_{i \in F} Z_i = \emptyset.$$
  Since $\mathcal{C}$ is totally ordered and F is finite, there is $i^* \in F$ such that $Z_{i^*} \subseteq Z_i$ for all $i \in F$, hence
  $$\bigcap_{i \in F} Z_i = Z_{i^*} \neq \emptyset$$
  since $Z_{i^*} \in \mathcal{Z}$.

Thus Zorn's lemma implies that $\mathcal{Z}$ has a minimal element, i.e., $\exists E \in \mathcal{Z}$ such that
$$\forall Z \in \mathcal{Z} \quad Z \subseteq E \Rightarrow Z = E.$$

It remains to show that $(E, T|_E)$ is minimal. But this follows immediately from the preceding proposition:
Suppose $Z \subseteq E$ is closed, non-empty, and $TZ = Z$. Then $Z \in \mathcal{Z}$, hence we have $Z = E$.

$\boxtimes$

Corollary (Birkhoff recurrence)

Let X compact metric and $T : X \to X$ a homeomorphism. Then (X, T) admits a recurrent point, i.e.,
$$\exists x \in X \quad x \in \omega^+(x).$$

Proof of the corollary:

Let $E \subseteq X$ closed non-empty s.t. $TE = E$ and s.t. $(E, T|_E)$ is minimal.
Let $x_0 \in E$ arbitrary. Note that we know that $O(x_0)$ is dense in E but this doesn't immediately tell us that x_0 is recurrent since $x_0 = T^0 x_0$. Still, this seems plausible if one considers that we should also return close to x_0.
We need to construct from x_0 a closed, T-invariant subset of E, then we can use minimality.

Claim:

$\omega^+(x_0)$ is T-invariant, closed, non-empty.

Non-empty: Since E is compact (as a closed subset of a compact space), we know that $(T^n x_0)_{n \in \mathbb{N}}$ has a convergent subsequence. Its limit is contained in $\omega^+(x_0)$.

Closed: Let $A = E - \omega^+(x_0)$. Then $y \in A \Leftrightarrow \exists r > 0 \; \exists n_0 \in \mathbb{N}$ s.t.
$$\forall n \geq n_0 \quad d(T^n x_0, y) \geq r.$$
Let $U = B_{r/2}(y)$. If $y' \in U$, then
$$\forall n \geq n_0 \quad d(T^n x_0, y') \geq \frac{r}{2},$$
thus $U \subseteq A$ and, hence, A is open.

T-invariant: Suppose $z \in T\omega^+(x_0)$. ~~Then~~ Let $y \in \omega^+(x_0)$ s.t. $z = Ty$. Let $(n_k)_{k \in \mathbb{N}}$ a sequence of natural numbers s.t. $n_k \nearrow \infty$ as $k \to \infty$ and s.t.
$$y = \lim_{k \to \infty} T^{n_k} x_0.$$
Then $z = Ty \underset{\text{continuity of } T}{=} \lim_{k \to \infty} T^{n_k + 1} x_0 \Rightarrow z \in \omega^+(x_0)$. Hence $T\omega^+(x_0) \subseteq \omega^+(x_0)$.

For the opposite inclusion, let $y \in \omega^+(x_0)$ and $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ s.t. $n_k \nearrow \infty$ and $T^{n_k}x_0 \xrightarrow{k\to\infty} y$. By compactness, the sequence

$$\left(T^{n_k - 1}x_0\right)_{k\in\mathbb{N}} \in X^{\mathbb{N}}$$

has a convergent subsequence $\left(T^{n_{k_\ell}-1}x_0\right)_{\ell\in\mathbb{N}}$, where $k_\ell \nearrow \infty$ as $\ell\to\infty$.

Then

$$T\Big(\underbrace{\lim_{\ell\to\infty} T^{n_{k_\ell}-1}x_0}_{\in\omega^+(x_0)}\Big) \overset{\text{continuity}}{=} \lim_{\ell\to\infty} T^{n_{k_\ell}}x_0 \overset{\text{convergence}}{=} \lim_{k\to\infty} T^{n_k}x_0 = y.$$

Hence $y \in T\omega^+(x_0)$, and therfore $\omega^+(x_0) \subseteq T\omega^+(x_0)$.

$\boxtimes$

Rmk: $d(x,y) = \min_{n\in\mathbb{Z}} |x+n-y|$

__Example:__ We can finally prove that R_α is minimal whenever $\alpha \notin \mathbb{Q}/\mathbb{Z}$.

To this end let $\varepsilon>0$ arbitrary. We claim that for $L = \left\lfloor \frac{1}{d(n\alpha,0)} \right\rfloor$

$$\forall x\in\mathbb{T}\ \exists n\in\mathbb{N}\ \mathcal{T}_L(x) = \{x + kn\alpha : 0 \le k \le L\}$$

is ε-dense, i.e., $\forall y\in\mathbb{T}\ \exists z\in \mathcal{T}_L(x)\quad d(y,z)<\varepsilon$.

Since x and ε arbitrary, this implies that every orbit is dense.

Now use Birkhoff recurrence and fix $x_0 \in \mathbb{T}$ recurrent for R_α. This x_0 will not play any particular role later on. Choose $n\in\mathbb{N}$ such that

$$d(R_\alpha^n x_0, x_0) < \varepsilon$$

$$\parallel$$

$$d(x_0 + n\alpha, x_0) = d(n\alpha, 0).$$

Since α is irrational, $n\alpha \neq 0$ and, hence, $d(n\alpha,0)>0$. Let

$$L = \left\lfloor \frac{1}{d(n\alpha,0)} \right\rfloor > \frac{1}{\varepsilon}.$$

We claim that $\forall y\in\mathbb{T}\ \exists k\in\mathbb{N}_0\cap[0,L]$ s.t.

$$d(y, x+kn\alpha) < \varepsilon.$$

It suffices to show that $d(y-x, kn\alpha)<\varepsilon$, i.e.

$$\forall y\in\mathbb{T}\ \exists 0\le |k| \le L\quad d(y, kn\alpha)<\varepsilon.$$

Let $y\in[0,1)$. ~~We partition~~ We assume that $\exists m\in\mathbb{Z}$ s.t. $\overset{\delta}{\overset{\parallel}{n\alpha+m}} \in [0,\varepsilon)$, i.e., that "$n\alpha$" is positive. The second case works similarly. Then

$$\cancel{[0,1) = \bigsqcup_{k=1}^{L-1} [kn\alpha+m, (k+1)n\alpha+m) \sqcup [0, n\alpha+m) \sqcup [Ln\alpha+m, 1).}$$

~~Note that it~~

Note that $k\delta = kn\alpha + km \equiv kn\alpha \mod 1$. Then

$$[0,1) = \bigsqcup_{k=1}^{L-1} [k\delta, (k+1)\delta) \sqcup [0,\delta) \sqcup [L\delta, 1),$$

where we use that

$$0 < L\delta = \left\lfloor \frac{1}{\delta} \right\rfloor \delta < 1,$$

since $\delta^{-1} \notin \mathbb{Z}$. We also note that

$$1 - L\delta = \underbrace{\left(\frac{1}{\delta} - \left\lfloor\frac{1}{\delta}\right\rfloor\right)}_{=\left\{\frac{1}{\delta}\right\}}\delta < \delta.$$

$\boxtimes$

__Remark:__ We have actually shown that $\{x + kn\alpha : 1\le k\le L\}$ is ε-dense for some $n\in\mathbb{N}$. Hence $\mathcal{O}^+(x)$ is dense.

__Corollary:__

Let $\varepsilon>0$, then $\forall x\in\mathbb{T}\ \exists\infty$-many $(p,q)\in\mathbb{Z}\times\mathbb{N}$ such that

$$\left|x - \frac{p}{q}\right| < \frac{\varepsilon}{q}.$$

__Proof:__ If $x\in\mathbb{Q}/\mathbb{Z}$, there is nothing to show.

~~Suppose $x\notin\mathbb{Q}/\mathbb{Z}$, then $\mathcal{O}^+(x_0)$ is dense for R_x. Thus for every $\varepsilon>0$ there exists $q\in\mathbb{N}$ such that~~

- Suppose $x\notin\mathbb{Q}/\mathbb{Z}$. We have seen that there are $n, L\in\mathbb{N}$ s.t.

$$\{knx;\ 1\le k\le L\} \subseteq \mathbb{T}$$

is ε-dense. Hence there exists $q = kn\in\mathbb{N}$ s.t.

$$\min_{p\in\mathbb{Z}} |qx - p| < \varepsilon$$

$\boxtimes$

__Ex:__ Find all minimal subsystems of R_α, $\alpha\in\mathbb{Q}/\mathbb{Z}$

__Ex:__ Show that $\omega^+(x)$ is non-empty, closed, T-invariant.

End of lecture 3