# I.2 Denjoy's theorem

In this section, we will discuss topological conjugacy, i.e., whether two homeomorphism of a given compact metric space lie in the same conjugacy class; note that Homeo(X) is a group! It turns out that, in fact, one would also like to compare homeomorphisms up to equivalence of the underlying topological space.

<u>Definition:</u>

Let $X_1, X_2$ be compact metric spaces and let $T_i \in \mathrm{Homeo}(X_i)$. The systems $(X_1, T_1)$ and $(X_2, T_2)$ are <u>topologically conjugate</u> if $\exists h: X_1 \longrightarrow X_2$ a homeomorphism such that

$$\begin{tikzcd} X_1 \arrow[r, "T_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[d, "h"] \\ X_2 \arrow[r, "T_2"'] & X_2 \end{tikzcd},$$

i.e.

$$h \circ T_1 = T_2 \circ h$$

<u>Remark:</u> Topological ~~equ~~ conjugacy is an equivalence relation.

Topological conjugacy captures whether two systems are essentially the same, e.g.

<u>Proposition</u> (without proof/exercise)

Let $X_1, X_2$ compact metric and $T_i \in \mathrm{Homeo}(X_i)$. Suppose that $(X_1, T_1)$ and $(X_2, T_2)$ are topologically conjugate. Then

(i) $(X_1, T_1)$ is topologically transitive $\iff$ $(X_2, T_2)$ is topologically transitive,

(ii) $(X_1, T_1)$ is minimal $\iff$ $(X_2, T_2)$ is minimal, and

(iii) $\mathrm{Fix}(T_2^n) = h(\mathrm{Fix}(T_1^n))$ for all $n \in \mathbb{Z}$.

In this section we will provide a special case of a classification, called Denjoy's theorem. More precisely, we are going to classify certain minimal homeomorphisms of the circle. We will require some preliminary notions.



## I.2.1 – Homeomorphisms of the circle and their rotation number

<u>Definition</u>

Let $T: \mathbb{T} \longrightarrow \mathbb{T}$ be a homeomorphism. A <u>lift</u> of $T$ is a map $\hat{T}: \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\hat{T}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \arrow[d, "\pi"] \\ \mathbb{T} \arrow[r, "T"'] & \mathbb{T} \end{tikzcd},$$

where $\pi: \mathbb{R} \longrightarrow \mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the canonical projection.

<u>Exercise:</u>

Let $T: \mathbb{T} \longrightarrow \mathbb{T}$ a homeomorphism. Let $y_0 \in \mathbb{R}$ such that

$$y_0 \bmod 1 = T(0 \bmod 1).$$

Then there exists a unique lift $\hat{T}: \mathbb{R} \longrightarrow \mathbb{R}$ such that $\hat{T}(0) = y_0$. Moreover, if $\tilde{T}: \mathbb{R} \to \mathbb{R}$ is a lift of $T$, then there is $k \in \mathbb{Z}$ such that

$$\forall t \in \mathbb{R} \quad \tilde{T}(t) = \hat{T}(t) + k.$$

Finally, $\hat{T}$ is a homeomorphism and $\forall n \in \mathbb{Z}$ $\hat{T}^n$ is a lift of $T^n$.

<u>Hint:</u> a) Let $\bar{\gamma}: [0,1] \xrightarrow{C^0} \mathbb{T}$ and let $y \in \mathbb{R}$ s.t. $y \bmod 1 = \bar{\gamma}(0)$. Then there exists a unique path $\gamma: [0,1] \xrightarrow{C^0} \mathbb{R}$ such that $\gamma(0) = y$ and $\bar{\gamma} = \pi \circ \gamma$.

b) Given $y \in \mathbb{R}$, let $\sigma_y: [0,1] \longrightarrow \mathbb{R}$ be the path given by $\sigma_y(t) = ty + (1-t)y_0$.

Define $\bar{\gamma}_y: [0,1] \xrightarrow{C^0} \mathbb{T}$ by $\bar{\gamma}_y(t) = T \circ \sigma_y(t)$ and let $\gamma_y: [0,1] \xrightarrow{C^0} \mathbb{R}$ be the lift given by $\gamma_y(0) = y_0$ and $\bar{\gamma}_y = \pi \circ \gamma_y$. Define

$$\forall y \in \mathbb{R} \quad \hat{T}(y) = \gamma_y(1).$$

<u>Lemma:</u>

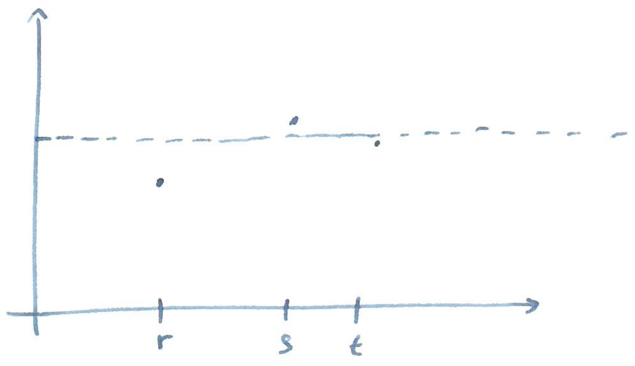
Let $\hat{T}: \mathbb{R} \longrightarrow \mathbb{R}$ be a homeomorphism (continuous and injective suffices). Then $\hat{T}$ is monotonic (strictly)

<u>Proof:</u> This follows from the intermediate value theorem. Suppose $\hat{T}$ is <u>not</u> monotonic. Then there are $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ distinct, $x_2 \neq y_2$ s.t. $x_i < y_i$ and

$$f(x_1) < f(y_1) \quad \wedge \quad f(x_2) > f(y_2).$$

By the intermediate value theorem, this reduces to (not even needed, just look at all the arrangements)

$$\exists r < s < t \text{ s.t. either } f(r) < f(s) \wedge f(s) > f(t),$$
$$\text{or } \quad f(r) > f(s) \wedge f(s) < f(t).$$



$\exists\, x \in (r,s), y \in (s,t)$ such that
$f(x) = f(y)$.
by the intermediate value theorem.

**Lemma**

Let $T : \mathbb{T} \longrightarrow \mathbb{T}$ be a homeomorphism s.t. any lift $\hat{T}$ is increasing, i.e., $T$ is orientation preserving. Then

$$\forall y \in \mathbb{R}\ \forall n \in \mathbb{Z}\ \forall k \in \mathbb{Z} \quad \hat{T}^n(y+k) = \hat{T}^n(y) + k$$

In particular

$$\forall y \in \mathbb{R}\ \forall n \in \mathbb{Z} \qquad \hat{T}^n(y) = \hat{T}^n(\{y\}) + \lfloor y \rfloor .$$

Proof: We start with

$$\hat{T}^n(y) \bmod 1 = \hat{T}^n(y+1) \bmod 1,$$

i.e., there exists $k \in \mathbb{Z}$ s.t.

$$k = \hat{T}^n(y+1) - \hat{T}^n(y).$$

Since $\hat{T}$ is monotonic, so is $\hat{T}^n$ and, hence $k \in \mathbb{N}$. Since $\hat{T}^n$ is a homeomorphism of the torus, we know that

$$\forall y \in \mathbb{R}\ \forall a \in (0,1) \quad \hat{T}^n(y \bmod 1) \neq \hat{T}^n(y+a \bmod 1),$$

i.e.,

$$\forall y \in \mathbb{R}\ \forall a \in (0,1) \quad \hat{T}^n(y) \bmod 1 \neq \hat{T}^n(y+a) \bmod 1.$$

Hence

$$\hat{T}^n(y+1) \leq \hat{T}^n(y) + 1$$

and, thus, $k = 1$, i.e., $\hat{T}^n(y) + 1 = \hat{T}^n(y+1)$.

The rest follows by induction. $\boxtimes$

**Lemma**

Let $T : \mathbb{T} \longrightarrow \mathbb{T}$ be an orientation preserving homeomorphism. Let $\hat{T} : \mathbb{R} \longrightarrow \mathbb{R}$ be a lift of $T$. If $k \in \mathbb{N}$ and $n \in \mathbb{Z}$, then

$$\forall y_1, y_2 \in \mathbb{R} \quad |y_1 - y_2| \leq k \Rightarrow |\hat{T}^n(y_1) - \hat{T}^n(y_2)| \leq k.$$

Proof: Suppose w.l.o.g. that $y_1 < y_2$. Then

$$\hat{T}(y_2) = \hat{T}(y_1 + \{y_2 - y_1\}) + \lfloor y_2 - y_1 \rfloor .$$

If $\{y_2 - y_1\} = 0$, then $\lfloor y_2 - y_1 \rfloor = |y_2 - y_1| \leq k$ and monotonicity implies

$$|\hat{T}(y_2) - \hat{T}(y_1)| = \hat{T}(y_2) - \hat{T}(y_1) \leq \lfloor y_2 - y_1 \rfloor \leq k.$$

So suppose that $\{y_2 - y_1\} > 0$, then

$$\hat{T}(y_1) < \hat{T}(y_1 + \{y_2 - y_1\}) < \hat{T}(y_1 + 1) = \hat{T}(y_1) + 1.$$

Therefore

$$\hat{T}(y_2) < \hat{T}(y_1) + 1 + \lfloor y_2 - y_1 \rfloor = \hat{T}(y_1) + \lceil y_2 - y_1 \rceil \leq \hat{T}(y_1) + k.$$

$\boxtimes$

Using this lemma, we can define the rotation number of a homeomorphism of the circle, which is the average rotation.

**Definition** (Rotation number)

Let $T : \mathbb{T} \longrightarrow \mathbb{T}$ be an orientation preserving homeomorphism. The rotation number of $T$ is defined as

$$\rho(T) = \limsup_{n \to \infty} \frac{\hat{T}^n(y)}{n} \bmod 1,$$

where $y \in \mathbb{R}$ is arbitrary and $\hat{T}$ is any lift of $T$.

**Remark:** • First question: is this well-defined? (if finite) Let $y_1, y_2 \in \mathbb{R}$ and $\hat{T}, \tilde{T}$ two lifts of $T$. Let $\ell \in \mathbb{Z}$ such that

$$\forall y \in \mathbb{R} \quad \tilde{T}(y) = \hat{T}(y) + \ell .$$

Then

$$\forall y \in \mathbb{R}\ \forall n \in \mathbb{Z} \quad \tilde{T}^n(y) = \hat{T}^n(y) + \ell n.$$

For example

$$\tilde{T}^2(y) = \tilde{T}(\tilde{T}(y)) = \tilde{T}(\hat{T}(y) + \ell) = \tilde{T}(\hat{T}(y)) + \ell = \hat{T}(\hat{T}(y)) + 2\ell$$
$$= \hat{T}^2(y) + 2\ell;$$

the general case then follows by induction.

Hence, for $n \in \mathbb{N}$, we have

$$\frac{\hat{T}^n(y_1)}{n} - \frac{\tilde{T}^n(y_2)}{n} = \frac{\hat{T}^n(y_1) - \hat{T}^n(y_2)}{n} + \frac{\hat{T}^n(y_2) - \tilde{T}^n(y_2)}{n}$$
$$= \ell + O\left(\frac{\lceil y_1 - y_2 \rceil}{n}\right) \longrightarrow 0 \bmod 1.$$

• Second question: Is this finite?

Lemma:

Let $T : \mathbb{T} \longrightarrow \mathbb{T}$ be an orientation preserving homeomorphism and let $\hat{T}$ be a lift of $T$. There exists for every $y \in \mathbb{R}$ some $C > 0$ depending on $\hat{T}$ and $y$ such that $\forall n \in \mathbb{N}$ $|\hat{T}^n(y)| \leq Cn$.

Proof: Recall that $\hat{T}^n(y) = \hat{T}^n(\{y\}) + \lfloor y \rfloor$. ~~Then~~ Let $y_0 = \hat{T}(0)$ and denote

$$g : \mathbb{R} \longrightarrow [y_0, y_0 + 1)$$
$$y \longmapsto g(y) = \hat{T}(\{y\}).$$

Then

$$\hat{T}^n(\{y\}) = g^n(y) + \sum_{k=1}^{n-1} \lfloor g^k(y) \rfloor.$$

Indeed,

$$\hat{T}^2(\{y\}) = \hat{T}(\hat{T}(\{y\})) = \hat{T}(\{\hat{T}(\{y\})\}) + \lfloor \hat{T}(\{y\}) \rfloor$$
$$= g^2(y) + \lfloor g(y) \rfloor$$

and induction yields

$$\hat{T}^{n+1}(\{y\}) = \hat{T}(\hat{T}^n(\{y\}))$$
$$= \hat{T}\Big(g^n(y) + \sum_{k=1}^{n-1} \lfloor g^k(y) \rfloor\Big) = \hat{T}(g^n(y)) + \sum_{k=1}^{n-1} \lfloor g^k(y) \rfloor$$
$$= \hat{T}(\{g^n(y)\}) + \lfloor g^n(y) \rfloor + \sum_{k=1}^{n-1} \lfloor g^k(y) \rfloor$$
$$= g^{n+1}(y) + \sum_{k=1}^{n} \lfloor g^k(y) \rfloor.$$

Now $g^k(y) \in [y_0, y_0 + 1)$ for all $k \in \mathbb{N}$, hence

$$\forall k \in \mathbb{N} \quad \lfloor g^k(y) \rfloor \in \{\lfloor y_0 \rfloor, \lfloor y_0 + 1 \rfloor\}.$$

Let $M > 0$ s.t. $M > \max\{|\lfloor y_0 \rfloor|, |\lfloor y_0 + 1 \rfloor|\}$. Then this implies that

$$|\hat{T}^n(\{y\}) - g^n(y)| \leq M(n-1).$$

In particular

$$\hat{T}^n(y) = \hat{T}^n(\{y\}) + \lfloor y \rfloor = O(n).$$
$$= \underbrace{\hat{T}^n(\{y\}) - g^n(y)}_{= O_{\hat{T}}(n)} + \underbrace{g^n(y) + \lfloor y \rfloor}_{= O_{\hat{T}, y}(1)}$$



Example: Let $\alpha \in \mathbb{R}$, then $\rho(R_\alpha) = \alpha \mod 1$.

Proof: Let $\hat{R}_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$
$$y \longmapsto y + \alpha.$$

Then $\hat{R}_\alpha$ is a lift of $R_\alpha$ and

$$\frac{\hat{R}_\alpha^n(y)}{n} = \frac{y}{n} + \alpha \longrightarrow \alpha \mod 1.$$

Note: In this case, the sequence $\frac{\hat{R}_\alpha^n(y)}{n}$ even converges, i.e., we don't need to worry about limsup vs. liminf. This is generally the case.

Proposition:

Let $T : \mathbb{T} \longrightarrow \mathbb{T}$ be an orientation preserving homeomorphism.

(i) $\rho(T) = \lim_{n \to \infty} \frac{\hat{T}^n(y)}{n} \mod 1$.

(ii) If $T$ has a periodic point, then $\rho(T) \in \mathbb{Q}/\mathbb{Z}$

(iii) If $T$ has no periodic point, then $\rho(T) \notin \mathbb{Q}/\mathbb{Z}$

(iv) $\forall n \in \mathbb{N}$ $\rho(T^n) = n\rho(T) \mod 1$.

Proof: (i) $\forall n \in \mathbb{Z}$ let $y_n := \hat{T}^n(0)$, so that by monotonicity

$$\forall y \in \mathbb{R} \quad \hat{T}^n(\{y\}) \in [y_n, y_n + 1) = \{y_n\} + [\lfloor y_n \rfloor, \lfloor y_n \rfloor + 1).$$

Hence

$$\forall y \in \mathbb{R} \quad \hat{T}^n(\{y\}) - \{y\} \in \underbrace{\{y_n\} - \{y\}}_{\in(-1,1)} + [\lfloor y_n \rfloor, \lfloor y_n \rfloor + 1) \subseteq [k_n - 2, k_n + 2],$$

where $k_n := \lfloor y_n \rfloor$. In particular, this shows that for all $n \in \mathbb{Z}$ there is $k_n \in \mathbb{Z}$ such that

$$\left| \frac{\hat{T}^n(0)}{n} - \frac{k_n}{n} \right| \leq \frac{2}{|n|}.$$

We will use this to prove that $\left(\frac{\hat{T}^n(0)}{n}\right)_{n \in \mathbb{N}}$ is Cauchy. Together with the earlier arguments, this implies that $\left(\frac{\hat{T}^n(y)}{n}\right)_{n \in \mathbb{N}}$ is Cauchy for any lift $\hat{T}$ and for any $y \in \mathbb{R}$.

Claim:

This term we can compare to both $\frac{k_n}{n}$ and $\frac{k_m}{m}$, which will allow us to "link" $\frac{\hat{T}^m(0)}{m}$ and $\frac{\hat{T}^n(0)}{n}$.

Let $m, n \in \mathbb{N}$, then $\left| \frac{\hat{T}^{nm}(0)}{nm} - \frac{k_n}{n} \right| \leq \frac{2}{n}$.

In order to prove the claim, note that (for $n, m \in \mathbb{N}$)

$$\hat{T}^{nm}(0) = \hat{T}^n(\hat{T}^{n(m-1)}(0)) - \hat{T}^{n(m-1)}(0)$$
$$+ \dots + \hat{T}^n(\hat{T}^n(0)) - \hat{T}^n(0) + \hat{T}^n(0) - 0$$
$$= \sum_{j=0}^{m-1} \left( \hat{T}^n(\hat{T}^{nj}(0)) - \hat{T}^{nj}(0) \right).$$

Note that for any $y \in \mathbb{R}$

$$\hat{T}^n(y) - y = \hat{T}^n(\{y\}) + \lfloor y \rfloor - \{y\} - \lfloor y \rfloor = \hat{T}^n(\{y\}) - \{y\},$$

hence

$$\hat{T}^{nm}(0) \in [m(k_n - 2), m(k_n + 2)],$$

Thus

$$-\frac{2}{n} \le \frac{\hat{T}^{nm}(0)}{mn} - \frac{k_n}{n} \le \frac{2}{n}$$

as desired.

It follows that

$$\left| \frac{\hat{T}^m(0)}{m} - \frac{\hat{T}^n(0)}{n} \right| \le \left| \frac{\hat{T}^m(0)}{m} - \frac{k_m}{m} \right| + \left| \frac{k_m}{m} - \frac{\hat{T}^{mn}(0)}{mn} \right| + \left| \frac{\hat{T}^{mn}(0)}{mn} - \frac{k_n}{n} \right| + \left| \frac{k_n}{n} - \frac{\hat{T}^n(0)}{n} \right|.$$

$$\le \frac{4}{m} + \frac{4}{n} \xrightarrow{n, m \to \infty} 0.$$

End of lecture 4.

(ii) Suppose $n \in \mathbb{N}$, $x \in \mathbb{T}$, and $T^n(x) = x$. Let $y \in \mathbb{R}$ s.t. $y = x \bmod 1$, then $\hat{T}^n(y) - y =: k \in \mathbb{Z}$.
Let $r \in \mathbb{N}$ arbitrary and write $n = pr + q$, where $0 \le q < n$. Then

$$\frac{\hat{T}^n(y)}{n} = \frac{\hat{T}^{pr+q}(y)}{n} = \frac{\hat{T}^q(\hat{T}^{pr}(y))}{n}.$$

Note that by induction

$$\hat{T}^r(y) = y + k,\ \hat{T}^{2r}(y) = \hat{T}^r(\hat{T}^r(y)) = \hat{T}^r(y+k) = \hat{T}^r(y) + k = y + 2k,$$
$$\dots, \hat{T}^{pr}(y) = y + pk.$$

Thus

$$\frac{\hat{T}^n(y)}{n} = \frac{\hat{T}^q(y+pk)}{n} = \frac{\hat{T}^q(y)}{n} + \frac{pk}{n} = \frac{\hat{T}^q(y)}{n} + \frac{p}{pr+q} \cdot k \longrightarrow \frac{k}{r},$$

thus $\rho(T) \in \mathbb{Q}/\mathbb{Z}$.

(iv) As $\hat{T}^n$ is a lift of $T^n$, we find that for all $n \in \mathbb{N}$:

$$\rho(T^n) = \lim_{k \to \infty} \frac{\hat{T}^{kn}(y)}{k} = n \cdot \lim_{k \to \infty} \frac{\hat{T}^{kn}(y)}{kn} = n \cdot \lim_{k \to \infty} \frac{\hat{T}^k(y)}{k} = n \rho(T).$$

For $\rho(T^{-1})$, note that $\hat{T}^n(\hat{T}^{-n}(0)) - \hat{T}^{-n}(0) = -\hat{T}^{-n}(0) \in [k_n - 2, k_n + 2]$, so w.l.o.g. $k_{-n} = -k_n$ $\therefore$ $\left| \frac{\hat{T}^{-n}(0)}{n} + \frac{\hat{T}^n(0)}{n} \right| \le \frac{4}{n}$ $\xrightarrow{} 0$

$\le \left| \frac{\hat{T}^{-n}(0)}{n} - \frac{k_{-n}}{n} \right| + \left| \frac{k_{-n}}{n} + \frac{\hat{T}^n(0)}{n} \right|$
$= \left| \frac{\hat{T}^{-n}(0)}{n} - \frac{k_n}{n} \right| + \left| \frac{k_n}{n} - \frac{\hat{T}^n(0)}{n} \right|$

(iii) Suppose for sake of contradiction that $\rho(T) = \frac{p}{q} \bmod 1$ for $p \in \mathbb{Z}, q \in \mathbb{N}$.
Let $S = T^q$. As shown, we have

$$\rho(S) = q \rho(T) \bmod 1 = 0 \bmod 1.$$

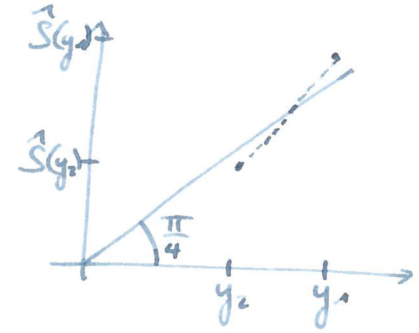
Note that $S$ has no fixed points since any fixed point for $S$ is a periodic point for $T$. Let $\hat{S}$ be a lift of $S$. Since $S$ is orientation preserving (since $\hat{T}^q$ is monotonic), $\hat{S}$ is strictly increasing. Shifting $\hat{S}$, we can assume $\lim_{n \to \infty} \frac{\hat{S}^n(0)}{n} = 0$.

Recall: $\tilde{T}(y) = \hat{T}(y) + \ell \Rightarrow \lim_{n \to \infty} \frac{\tilde{T}^n(0)}{n} = \ell + \lim_{n \to \infty}$ …

Claim:

Either $\forall y \in \mathbb{R}\ \hat{S}(y) > y$ or $\forall y \in \mathbb{R}\ \hat{S}(y) < y$.

Pf of claim: Assume otherwise, i.e., suppose that $\exists y_1, y_2 \in \mathbb{R}$ s.t. $y_1 < \hat{S}(y_1)$, $\hat{S}(y_2) < y_2$.
Then one of the tuples $(y_i, \hat{S}(y_i)) \in \mathbb{R}^2$ lies below and one above the slope 1 line in $\mathbb{R}^2$:




By the intermediate value theorem, there is $y \in (\min\{y_1, y_2\}, \max\{y_1, y_2\})$ such that $y = \hat{S}(y)$. This is absurd.

Case 1: $\forall y \in \mathbb{R}\ \hat{S}(y) > y$.

In particular $\hat{S}(0) > 0$. Suppose $\exists k \in \mathbb{N}$ s.t. $\hat{S}^k(0) \ge 1$. Then

$$\hat{S}^{mk}(0) = \hat{S}^k(\hat{S}^{(m-1)k}(0)) \underset{\text{induction}}{\ge} \hat{S}^k(m-1) = \hat{S}^k(0) + (m-1) \ge m,$$

so $\hat{S}^{mk}(0) \ge m$ for all $m \in \mathbb{N}$. Thus

$$\lim_{n \to \infty} \frac{\hat{S}^n(0)}{n} \ge \lim_{m \to \infty} \frac{\hat{S}^{mk}(0)}{mk} \ge \frac{1}{k}.$$

It follows that $\forall k \in \mathbb{N}\ \hat{S}^k(0) < 1$. Hence, since $\hat{S}^{k+1}(0) > \hat{S}^k(0)$ for all $k \in \mathbb{N}$, by Weierstrass the limit $y^* = \lim_{k \to \infty} \hat{S}^k(0)$ exists. But

$$\hat{S}(y^*) = \hat{S}(\lim_{k \to \infty} \hat{S}^k(0)) = \lim_{k \to \infty} \hat{S}^{k+1}(0) = y^* \ ⚡$$