In order to prove the claim, note that (for $n, m \in \mathbb{N}$)

$$\hat{T}^{nm}(0) = \hat{T}^n\left(\hat{T}^{n(m-1)}(0)\right) - \hat{T}^{n(m-1)}(0)$$
$$+ \dots + \hat{T}^n(\hat{T}^n(0)) - \hat{T}^n(0) + \hat{T}^n(0) - 0$$
$$= \sum_{j=0}^{m-1} \left(\hat{T}^n(\hat{T}^{nj}(0)) - \hat{T}^{nj}(0)\right).$$

Note that for any $y \in \mathbb{R}$

$$\hat{T}^n(y) - y = \hat{T}^n(\{y\}) + \lfloor y \rfloor - \{y\} - \lfloor y \rfloor = \hat{T}^n(\{y\}) - \{y\},$$

hence

$$\hat{T}^{nm}(0) \in [m(k_n - 2), m(k_n + 2)],$$

thus

$$-\frac{2}{n} \leq \frac{\hat{T}^{nm}(0)}{mn} - \frac{k_n}{n} \leq \frac{2}{n}$$

as desired.

It follows that

$$\left|\frac{\hat{T}^m(0)}{m} - \frac{\hat{T}^n(0)}{n}\right| \leq \left|\frac{\hat{T}^m(0)}{m} - \frac{k_m}{m}\right| + \left|\frac{k_m}{m} - \frac{\hat{T}^{mn}(0)}{mn}\right| + \left|\frac{\hat{T}^{mn}(0)}{mn} - \frac{k_n}{n}\right| + \left|\frac{k_n}{n} - \frac{\hat{T}^n(0)}{n}\right|$$

$$\leq \frac{4}{m} + \frac{4}{n} \xrightarrow{n,m \to \infty} 0.$$

End of lecture 4.

(ii) Suppose $m \in \mathbb{N}$, $x \in \mathbb{T}$, and $T^m(x) = x$. Let $y \in \mathbb{R}$ s.t. $y = x \mod 1$, then $\hat{T}^m(y) - y =: k \in \mathbb{Z}$.
Let $n \in \mathbb{N}$ arbitrary and write $n = pr + q$, where $0 \leq q < n$. Then

$$\frac{\hat{T}^n(y)}{n} = \frac{\hat{T}^{pr+q}(y)}{n} = \frac{\hat{T}^q(\hat{T}^{pr}(y))}{n}.$$

Note that by induction

$$\hat{T}^r(y) = y + k,\ \hat{T}^{2r}(y) = \hat{T}^r(\hat{T}^r(y)) = \hat{T}^r(y+k) = \hat{T}^r(y) + k = y + 2k,$$
$$\dots, \hat{T}^{pr}(y) = y + pk.$$

Thus

$$\frac{\hat{T}^n(y)}{n} = \frac{\hat{T}^q(y+pk)}{n} = \frac{\hat{T}^q(y)}{n} + \frac{pk}{n} = \frac{\hat{T}^q(y)}{n} + \frac{p}{pr+q} \cdot k \longrightarrow \frac{k}{r},$$

thus $\varrho(T) \in \mathbb{Q}/\mathbb{Z}$.

(iv) As $\hat{T}^n$ is a lift of T^n, we find that for all $n \in \mathbb{N}$:

$$\varrho(T^n) = \lim_{k\to\infty} \frac{\hat{T}^{kn}(y)}{k} = n \cdot \lim_{k\to\infty} \frac{\hat{T}^{kn}(y)}{kn} = n \cdot \lim_{k\to\infty} \frac{\hat{T}^k(y)}{k} = n\varrho(T).$$

For $\varrho(T^{-1})$, note that $\hat{T}^n(\hat{T}^{-n}(0)) - \hat{T}^{-n}(0) = -\hat{T}^{-n}(0) \in [k_n - 2, k_n + 2]$, so w.l.o.g. $k_{-n} = -k_n$ $\therefore$ $\left|\frac{\hat{T}^{-n}(0)}{n} + \frac{\hat{T}^n(0)}{n}\right| \leq \frac{4}{n}$ $\overset{8}{\downarrow}$

$$\leq \left|\frac{\hat{T}^{-n}(0)}{n} - \frac{k_{-n}}{n}\right| + \left|\frac{k_{-n}}{n} + \frac{\hat{T}^n(0)}{n}\right|$$
$$= \left|\frac{\hat{T}^{-n}(0)}{n} - \frac{k_n}{n}\right| + \left|\frac{k_n}{n} - \frac{\hat{T}^n(0)}{n}\right|$$

(iii) Suppose for sake of contradiction that $\varrho(T) = \frac{p}{q} \mod 1$ for $p \in \mathbb{Z}, q \in \mathbb{N}$.
Let $S = T^q$. As shown, we have

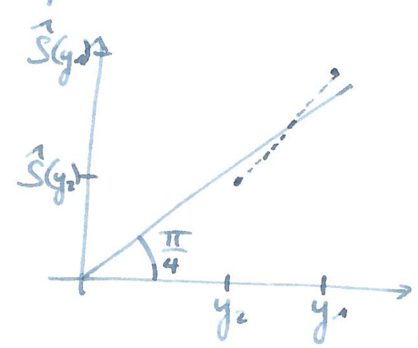
$$\varrho(S) = q\varrho(T) \mod 1 = 0 \mod 1.$$

Note that S has no fixed points since any fixed point for S is a periodic point for T. Let $\hat{S}$ be a lift of S. Since S is orientation preserving (since $\hat{T}^q$ is monotonic), $\hat{S}$ is strictly increasing. Shifting $\hat{S}$, we can assume $\lim_{n\to\infty} \frac{\hat{S}^n(0)}{n} = 0$.

Recall: $\tilde{T}(y) = \hat{T}(y) + l \Rightarrow \lim_{n\to\infty} \frac{\tilde{T}^n(0)}{n} = l + \lim_{n\to\infty}$

Claim:
Either $\forall y \in \mathbb{R}\ \hat{S}(y) > y$ or $\forall y \in \mathbb{R}\ \hat{S}(y) < y$.

Pf of claim: Assume otherwise, i.e., suppose that $\exists y_1, y_2 \in \mathbb{R}$ s.t. $y_1 < \hat{S}(y_1)$, $\hat{S}(y_2) < y_2$.
Then one of the tuples $(y_i, \hat{S}(y_i)) \in \mathbb{R}^2$ lies below and one above the slope 1 line in $\mathbb{R}^2$:



By the intermediate value theorem, there is $y \in (\min\{y_1, y_2\}, \max\{y_1, y_2\})$ such that $y = \hat{S}(y)$. This is absurd.

Case 1: $\forall y \in \mathbb{R}\ \hat{S}(y) > y$.
In particular $\hat{S}(0) > 0$. Suppose $\exists k \in \mathbb{N}$ s.t. $\hat{S}^k(0) \geq 1$. Then

$$\hat{S}^{mk}(0) = \hat{S}^k(\hat{S}^{(m-1)k}(0)) \underset{\text{induction}}{\geq} \hat{S}^k(m-1) = \hat{S}^k(0) + (m-1) \geq m,$$

so $\hat{S}^{mk}(0) \geq m$ for all $m \in \mathbb{N}$. Thus

$$\lim_{n\to\infty} \frac{\hat{S}^n(0)}{n} \geq \lim_{m\to\infty} \frac{\hat{S}^{mk}(0)}{mk} \geq \frac{1}{k}.$$

It follows that $\forall k \in \mathbb{N}\ \hat{S}^k(0) < 1$. Hence, since $\hat{S}^{k+1}(0) > \hat{S}^k(0)$ for all $k \in \mathbb{N}$, by Weierstrass the limit $y^* = \lim_{k\to\infty} \hat{S}^k(0)$ exists. But

$$\hat{S}(y^*) = \hat{S}(\lim_{k\to\infty} \hat{S}^k(0)) = \lim_{k\to\infty} \hat{S}^{k+1}(0) = y^*\ \lightning$$

<u>Case 2:</u> For all $y \in \mathbb{R}$ $\hat{S}(y) < y$.

Note that $\forall y \in \mathbb{R}$ $y < \hat{S}^{-1}(y)$ and $\rho(\hat{S}^{-1}) = -\rho(\hat{S}) = 0$. The preceding argument yields that $\hat{S}^{-1}$ has a fixed point, which is equivalent to $\hat{S}$ having a fixed point and, therefore, absurd.

$\blacksquare$

Now these are some useful properties of rotation numbers. What we will do is to classify the minimal homeomorphisms on $\mathbb{T}$ by their rotation number. Note that the preceding proposition shows that $T : \mathbb{T} \longrightarrow \mathbb{T}$ minimal $\Rightarrow \rho(T) \notin \mathbb{Q}/\mathbb{Z}$. Next time, we start the proof of

<u>Theorem</u> (Denjoy's theorem)

Let $T : \mathbb{T} \longrightarrow \mathbb{T}$ a minimal, orientation preserving homeomorphism and let $\rho = \rho(T)$. Then $(\mathbb{T}, T)$ and $(\mathbb{T}, R_\rho)$ are conjugate.

The disadvantag of this theorem is that it is a priori hard to decide whether a homeomorphism is minimal. So after the proof, we will give sufficient (and necessary) analytic conditions on T to guarantee minimality.

For the proof of Denjoy's theorem, the following will be useful.

<u>Lemma</u>:

Suppose $\rho(T) \notin \mathbb{Q}/\mathbb{Z}$.

(i) Let $n_1, n_2, m_1, m_2 \in \mathbb{Z}$, $x, y \in \mathbb{R}$. Then
$$\hat{T}^{n_1}(x) + m_1 < \hat{T}^{n_2}(x) + m_2 \Rightarrow \hat{T}^{n_1}(y) + m_1 < \hat{T}^{n_2}(y) + m_2.$$

(ii) The map
$$\Omega = \{n\rho(T) + m : n, m \in \mathbb{Z}\} \longrightarrow \Lambda = \{\hat{T}^n(0) + m : n, m \in \mathbb{Z}$$
$$n\rho(T) + m \longmapsto \hat{T}^n(0) + m$$
is well-defined for any choice of $\rho(T)$. Moreover, it preserves the natural ordering on $\mathbb{R}$.

<u>Proof:</u> (i) Suppose $\hat{T}^{n_2}(y) + m_2 < \hat{T}^{n_1}(y) + m_1$. Then certainly $n_1 \neq n_2$ and
$$\hat{T}^{n_1}(x) + m_1 - \hat{T}^{n_2}(x) - m_2 < 0 < \hat{T}^{n_1}(y) + m_1 - \hat{T}^{n_2}(y) - m_2.$$
Using the intermediate value theorem, there is $z \in \mathbb{R}$ s.t.
$$\hat{T}^{n_1}(z) \equiv \hat{T}^{n_2}(z) \mod \mathbb{Z}$$
$\Rightarrow T$ has a periodic point z mod $\mathbb{Z}$ ↯.

(ii) We first show that the map is well-defined. To this end note that
$$\mathbb{Z}^2 \longrightarrow \Omega,$$
$$(n, m) \longmapsto n\rho(T) + m$$
is a group homomorphism. Moreover, if $\ker \neq \{0\}$, then there is $(n, m) \in \mathbb{Z}^2 - \{0\}$ such that $n\rho(T) = -m$. Since $(n, m) \neq 0$ and $\rho(T) \notin \mathbb{Q}$, it follows
$$\rho(T) = -\frac{m}{n} \in \mathbb{Q} \text{ ↯}$$
Thus the map $(n, m) \longmapsto n\rho(T) + m$ is injective and, hence, the map
$$n\rho(T) + m \longmapsto \hat{T}^n(0) + m$$
is well-defined.

<u>Claim:</u>

The map is injective.

Suppose $\hat{T}^{n_1}(0) + m_1 = \hat{T}^{n_2}(0) + m_2$
$$\Rightarrow \hat{T}^{n_1}(0) \equiv \hat{T}^{n_2}(0) \mod \mathbb{Z}$$
$$\overset{\rho(T) \notin \mathbb{Q}}{\Rightarrow} n_1 = n_2 \Rightarrow m_1 = m_2.$$

<u>Claim:</u>

$n_1\rho(T) + m_1 < n_2\rho(T) + m_2 \Longleftrightarrow \hat{T}^{n_1}(0) + m_1 < \hat{T}^{n_2}(0) + m_2$.

"$\Longleftarrow$" suffices since the map is a bijection. So suppose
$$\hat{T}^{n_1}(0) + m_1 < \hat{T}^{n_2}(0) + m_2.$$
$$\|$$
$$\hat{T}^{n_1 - n_2}(\hat{T}^{n_2}(0)) + m_1$$
$$\overset{(i)}{\Rightarrow} \hat{T}^{n_1 - n_2}(0) + m_1 < m_2 \quad (\text{using } x = \hat{T}^{n_2}(0), y = 0).$$
$$\Leftrightarrow \hat{T}^{n_1 - n_2}(0) < m_2 - m_1$$
$$\overset{(i)}{\Rightarrow} \hat{T}^{2(n_1 - n_2)}(0) < \hat{T}^{n_1 - n_2}(0) + m_2 - m_1 < 2(m_2 - 1) \quad (\text{using } y = \hat{T}^{n_1 - n_2}(0),$$
$x = 0$, $n_1 = n_1 - n_2$, and $n_2 = 0$, $m_1 = 0$, $m_2 = m_2 - m_1$)

Similarly, if $\hat{T}^{k(n_1 - n_2)}(0) < \hat{T}^{(k-1)(n_1 - n_2)}(0) + m_2 - m_1$, then
$$\hat{T}^{(k+1)(n_1 - n_2)}(0) = \hat{T}^{k(n_1 - n_2)}(\hat{T}^{n_1 - n_2}(0)) \overset{(i)}{<} \hat{T}^{(k-1)(n_1 - n_2)}(\underbrace{*}_{\hat{T}^{n_1 - n_2}(0)}) + m_2 - m_1$$
$$= \hat{T}^{k(n_1 - n_2)}(0) + m_2 - m_1$$

It follows that

$$\forall k \geq 1 \quad \hat{T}^{k(n_1-n_2)}(0) < k(m_2-m_1).$$

Hence, letting $e = \operatorname{sign}(n_1-n_2)$, we have

$$e\rho(T) = \rho(T^e) = \lim_{n\to\infty} \frac{\hat{T}^{en}(0)}{n} = \lim_{k\to\infty} \frac{\hat{T}^{k(n_1-n_2)}(0)}{ek(n_1-n_2)}$$

$$\leq \frac{m_2-m_1}{e(n_1-n_2)}.$$

Note: $\rho(T) \notin \mathbb{Q} \Rightarrow \rho(T) \neq \frac{m_2-m_1}{n_1-n_2}$, thus

$$\rho(T)(n_1-n_2) = \rho(T)e^2(n_1-n_2) < m_2-m_1$$

$$\Rightarrow n_1\rho(T)+m_1 < n_2\rho(T)+m_2.$$

End of lecture 5

I.2.1 - Proof of Denjoy's theorem

Recall theorem ("Topological Denjoy")

If $T: \mathbb{T} \longrightarrow \mathbb{T}$ is minimal, orientation preserving homeomorphism with rotation number $\rho(T) = \rho \notin \mathbb{Q}$. Then T is conjugate to R_ρ, i.e., $\exists h: \mathbb{T} \to \mathbb{T}$ such that

$$\begin{CD} \mathbb{T} @>T>> \mathbb{T} \\ @VhVV @VVhV \\ \mathbb{T} @>>R_\rho> \mathbb{T} \end{CD}$$

Proof: Let $\hat{T}: \mathbb{R} \to \mathbb{R}$ be a lift. Since $\rho \notin \mathbb{Q}$, we know that

$$\Omega = \{n\rho + m : n, m \in \mathbb{Z}\} \subseteq \mathbb{R}$$

is dense. Since T is minimal, $\mathcal{O}(0) \subseteq \mathbb{T}$ is dense

$$\Rightarrow \Lambda = \{\hat{T}^n(0) + m : n, m \in \mathbb{Z}\} \subseteq \mathbb{R} \text{ is dense.}$$

Denote

$$\phi: \Lambda \longrightarrow \Omega$$
$$\hat{T}^n(0)+m \longmapsto n\rho + m.$$

By the preceding lemma, ϕ is an order preserving bijection between dense subsets of $\mathbb{R}$. Hence ϕ extends to a homeomorphism of $\mathbb{R}$.

Step 1: If $(n_k, m_k) \in \mathbb{Z}^2$ s.t. $x_k := \hat{T}^{n_k}(0) + m_k \xrightarrow{k\to\infty} x$, then

$$\phi(x) := \lim_{k\to\infty} \phi(x_k)$$

exists.

Let $a_1, b_1, a_2, b_2 \in \mathbb{Z}$ s.t.

$$\forall k \in \mathbb{N} \quad \hat{T}^{a_1}(0) + b_1 \leq x_k \leq \hat{T}^{a_2}(0) + b_2,$$

then

$$\forall k \in \mathbb{N} \quad a_1\rho + b_1 \leq \phi(x_k) \leq a_2\rho + b_2.$$

Step 2: The limit is independent of the choice of the sequence.

Given $x_k' \longrightarrow x$, define $\tilde{x}_k = \begin{cases} x_{k/2} & \text{if } 2 \mid k, \\ x'_{(k-1)/2} & \text{else.} \end{cases}$

A rmk. on step 1: Let $(x_{k_\ell})_{\ell\in\mathbb{N}}$ s.t. $y = \lim_{\ell\in\mathbb{N}} \phi(x_{k_\ell})$ exists. Let $\varepsilon > 0$ arbitrary and let $a_1(\varepsilon), a_2(\varepsilon), b_1(\varepsilon), b_2(\varepsilon) \in \mathbb{Z}$ s.t.

$$y - \varepsilon < a_1(\varepsilon)\rho + b_1(\varepsilon) + \tfrac{\varepsilon}{2} < y < a_2(\varepsilon)\rho + b_2(\varepsilon) - \tfrac{\varepsilon}{2} < y + \varepsilon$$

Let $L_\varepsilon \in \mathbb{N}$ s.t.

$$\ell > L_\varepsilon \Rightarrow a_1(\varepsilon)\rho + b_1(\varepsilon) + \tfrac{\varepsilon}{2} < \phi(x_{k_\ell}) < a_2(\varepsilon)\rho + b_2(\varepsilon) - \tfrac{\varepsilon}{2}$$
$$\Rightarrow \hat{T}^{a_1(\varepsilon)}(0) + b_1(\varepsilon) < x_{k_\ell} < \hat{T}^{a_2(\varepsilon)}(0) + b_2(\varepsilon).$$

$\therefore \exists K_\varepsilon \in \mathbb{N}$ s.t.

$$k > K_\varepsilon \Rightarrow \hat{T}^{a_1(\varepsilon)}(0) + b_1(\varepsilon) - \tfrac{\varepsilon}{2} < x_k < \hat{T}^{a_2(\varepsilon)}(0) + b_2(\varepsilon) + \tfrac{\varepsilon}{2}.$$

...

Claim:

$$\begin{CD} \mathbb{R} @>\hat{T}>> \mathbb{R} \\ @V\phi VV @VV\phi V \\ \mathbb{R} @>>\tilde{R}_\rho> \mathbb{R} \end{CD},$$

where $\tilde{R}_\rho(0) = \rho$.

It suffices to prove the claim on Λ. Let $n, m \in \mathbb{Z}$, then

$$\phi \circ \hat{T}(\hat{T}^n(0)+m) = \phi(\hat{T}^{n+1}(0)+m) = (n+1)\rho + m = n\rho + m + \rho = \tilde{R}_\rho(\phi(\hat{T}^n(0)+m)).$$