It follows that

$$\forall k \geq 1 \quad \hat{T}^{k(n_1-n_2)}(0) < k(m_2-m_1).$$

Hence, letting $e = \operatorname{sign}(n_1-n_2)$, we have

$$e\rho(T) = \rho(T^e) = \lim_{n\to\infty} \frac{\hat{T}^{en}(0)}{n} = \lim_{k\to\infty} \frac{\hat{T}^{k(n_1-n_2)}(0)}{ek(n_1-n_2)}$$

$$\leq \frac{m_2-m_1}{e(n_1-n_2)}.$$

Note: $\rho(T) \notin \mathbb{Q} \Rightarrow \rho(T) \neq \frac{m_2-m_1}{n_1-n_2}$, thus

$$\rho(T)(n_1-n_2) = \rho(T)e^2(n_1-n_2) < m_2-m_1$$

$$\Rightarrow n_1\rho(T)+m_1 < n_2\rho(T)+m_2.$$

End of lecture 5 ⊠

I.2.1 - Proof of Denjoy's theorem

Recall theorem ("Topological Denjoy")

If $T:\mathbb{T}\longrightarrow\mathbb{T}$ is minimal, orientation preserving homeomorphism with rotation number $\rho(T)=\rho\notin\mathbb{Q}$. Then T is conjugate to R_ρ, i.e., $\exists\, h:\mathbb{T}\to\mathbb{T}$ such that

$$\begin{tikzcd} \mathbb{T} \arrow[r, "T"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{T} \arrow[d, "h"] \\ \mathbb{T} \arrow[r, "R_\rho"'] & \mathbb{T} \end{tikzcd}$$

Proof: Let $\hat{T}:\mathbb{R}\to\mathbb{R}$ be a lift. Since $\rho\notin\mathbb{Q}$, we know that

$$\Omega = \{n\rho+m : n,m\in\mathbb{Z}\} \subseteq \mathbb{R}$$

is dense. Since T is minimal, $\mathcal{O}(0)\subseteq\mathbb{T}$ is dense

$$\Rightarrow \Lambda = \{\hat{T}^n(0)+m : n,m\in\mathbb{Z}\} \subseteq \mathbb{R} \text{ is dense.}$$

Denote

$$\phi:\Lambda\longrightarrow\Omega$$
$$\hat{T}^n(0)+m \longmapsto n\rho+m.$$

By the preceding lemma, ϕ is an order preserving bijection between dense subsets of $\mathbb{R}$. Hence ϕ extends to a homeomorphism of $\mathbb{R}$.

⌜Step 1: If $(n_k,m_k)\in\mathbb{Z}^2$ s.t. $x_k := \hat{T}^{n_k}(0)+m_k \xrightarrow{k\to\infty} x$, then

$$\phi(x) := \lim_{k\to\infty}\phi(x_k)$$

exists.

⌜Let $a_1,b_1,a_2,b_2\in\mathbb{Z}$ s.t.

$$\forall k\in\mathbb{N} \quad \hat{T}^{a_1}(0)+b_1 \leq x_k \leq \hat{T}^{a_2}(0)+b_2,$$

then

$$\forall k\in\mathbb{N} \quad a_1\rho+b_1 \leq \phi(x_k) \leq a_2\rho+b_2.$$ ⌟

Step 2: The limit is independent of the choice of the sequence.

⌜Given $x_k' \longrightarrow x$, define $\tilde{x}_k = \begin{cases} x_{k/2} & \text{if } 2|k, \\ x'_{(k-1)/2} & \text{else.}\end{cases}$ ⌟

A mug. on step 1: let $(x_{k_\ell})_{\ell\in\mathbb{N}}$ s.t. $y = \lim_{\ell\in\mathbb{N}}\phi(x_{k_\ell})$ exists. Let $\varepsilon>0$ arbitrary and let $a_1(\varepsilon),a_2(\varepsilon),b_1(\varepsilon),b_2(\varepsilon)\in\mathbb{Z}$ s.t.

$$y-\varepsilon < a_1(\varepsilon)\rho+b_1(\varepsilon)+\tfrac{\varepsilon}{2} < y < a_2(\varepsilon)\rho+b_2(\varepsilon)-\tfrac{\varepsilon}{2} < y+\varepsilon$$

Let $L_\varepsilon\in\mathbb{N}$ s.t.

$$\ell > L_\varepsilon \Rightarrow a_1(\varepsilon)\rho+b_1(\varepsilon)+\tfrac{\varepsilon}{2} < \phi(x_{k_\ell}) < a_2(\varepsilon)\rho+b_2(\varepsilon)-\tfrac{\varepsilon}{2}$$

$$\Rightarrow \hat{T}^{a_1(\varepsilon)}(0)+b_1(\varepsilon) < x_{k_\ell} < \hat{T}^{a_2(\varepsilon)}(0)+b_2(\varepsilon).$$

$\therefore \exists K_\varepsilon\in\mathbb{N}$ s.t.

$$k > K_\varepsilon \Rightarrow \hat{T}^{a_1(\varepsilon)}(0)+b_1(\varepsilon)-\tfrac{\varepsilon}{2} < x_k < \hat{T}^{a_2(\varepsilon)}(0)+b_2(\varepsilon)+\tfrac{\varepsilon}{2}.$$

...

⌟

Claim:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\hat{T}"] \arrow[d, "\phi"'] & \mathbb{R} \arrow[d, "\phi"] \\ \mathbb{R} \arrow[r, "\tilde{R}_\rho"'] & \mathbb{R} \end{tikzcd},$$

where $\tilde{R}_\rho(0)=\rho$.

It suffices to prove the claim on Λ. Let $n,m\in\mathbb{Z}$, then

$$\phi\circ\hat{T}(\hat{T}^n(0)+m) = \phi(\hat{T}^{n+1}(0)+m) = (n+1)\rho+m = n\rho+m+\rho = \tilde{R}_\rho(\phi(\hat{T}^n(0)+m)).$$

Claim

ϕ descends to a homeo of $\mathbb{R}/\mathbb{Z} = \mathbb{T}$:

$\forall x \in \mathbb{R} \quad \phi(x+1) = \phi(x) + 1$

Pf of claim: Again, it suffices to prove this on Λ:

$$\phi(\hat{T}^n(0) + m + 1) = n\varrho + (m+1) = (n\varrho + m) + 1 = \phi(\hat{T}^n(0) + m) + 1.$$

Claim:

$\Phi : \mathbb{T} \longrightarrow \mathbb{T}$

$x + \mathbb{Z} \longmapsto \phi(x) + \mathbb{Z}$

is a homeo.

Proof: Exercise $(U \subseteq \mathbb{T}$ open $\Leftrightarrow \pi^{-1}(U) \subseteq \mathbb{R}$ open

$\Rightarrow \phi(\pi^{-1}(U)) \subseteq \mathbb{R}$ open

$\Rightarrow \pi(\phi(\pi^{-1}(U))) \subseteq \mathbb{T}$ open$)$

That Φ is a bijection, can be seen like this:

$\Phi(x \bmod \mathbb{Z}) = \Phi(y \bmod \mathbb{Z})$

$\Leftrightarrow \phi(x) - \phi(y) \in \mathbb{Z}$.

Hence

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "T"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{T} \arrow[d, "\Phi"] \\
\mathbb{T} \arrow[r, "R_\varrho"'] & \mathbb{T}
\end{tikzcd}$$

$\square$

: How do you check that T is minimal? We'll give an analytic criterion.

Definition ("logarithmic total variation")

Let $T : \mathbb{T} \longrightarrow \mathbb{T}$ C^1. Define

$$\mathrm{Var}(\log|T'|) := \sup\left\{ \sum_{i=0}^{n-1} \left| \log|T'(x_{i+1})| - \log|T'(x_i)| \right| : 0 = x_0 < x_1 < \dots < x_n = 1, n \in \mathbb{N} \right\}$$

i.e., the total variation of $\log|T'|$.

Definition:

$T : \mathbb{T} \longrightarrow \mathbb{T} \in C^1$ has bounded logarithmic total variation if $\mathrm{Var}(\log|T'|) < \infty$.

Remark: Suppose that $T \in \mathrm{Homeo}(\mathbb{T}) \cap C^2(\mathbb{T})$. Then T has bounded logarithmic total variation.

Theorem (Denjoy)

If $T \in \mathrm{Homeo}(\mathbb{T}) \cap C^1(\mathbb{T})$ is orientation preserving and of bounded logarithmic total variation, if $\varrho(T) \notin \mathbb{Q}/\mathbb{Z}$, then T is topologically conjugate to $R_{\varrho(T)}$.

Note: We only have to show that T is minimal.

Lemma

Suppose $T \in \mathrm{Homeo}(\mathbb{T}) \cap C^1(\mathbb{T})$, $\varrho(T) \notin \mathbb{Q}/\mathbb{Z}$, and suppose there exist $C > 0$ and a sequence $(q_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ s.t. $q_n \xrightarrow{n \to \infty} \infty$ and

$$\forall x \in \mathbb{T} \qquad |(T^{q_n})'(x)|\,|(T^{-q_n})'(x)| \geq C.$$

Then T is minimal.

Proof: Suppose T is not minimal and choose $x \in \mathbb{T}$ s.t. $Y := \overline{\mathcal{O}(x)} \neq \mathbb{T}$. Since Y is closed, the complement contains a maximal open sub-interval $I_0 := (a,b) + \mathbb{Z}$.

Maximality implies that $a, b \in Y$.

Given $n \in \mathbb{Z}$, let $I_n := T^{-n} I_0$. We claim that all these intervals are maximal and pairwise disjoint.

For maximality, note that Y is (strongly) T-invariant, thus

$$\forall n \in \mathbb{Z} \quad I_n \subseteq \mathbb{T} - Y,$$

and necessarily T maps maximal subintervals to maximal subintervals, hence I_n is maximal.

For disjointness, let $m, n \in \mathbb{Z}$ s.t. $I_m \cap I_n = T^{-m} I_0 \cap T^{-n} I_0 \neq \emptyset$. By maximality, we have that $I_m = I_n$, hence $T^{-m} a = T^{-n} a$. Since $\varrho(T) \notin \mathbb{Q}/\mathbb{Z}$, it follows that $m = n$.

Let $|I_n|$ be the (un-)length of I_n. Then

$$\sum_{n \in \mathbb{Z}} |I_n| = \mathrm{vol}\left(\bigcup_{n \in \mathbb{Z}} I_n \right) \leq 1.$$

Hence $\lim_{|n| \to \infty} |I_n| = 0$. But

$$|I_{q_n}| + |I_{-q_n}| = \int_{T^{q_n} a}^{T^{q_n} b} 1\,dx + \int_{T^{-q_n} a}^{T^{-q_n} b} 1\,dx = \int_a^b \left(|(T^{q_n})'(x)| + |(T^{-q_n})'(x)| \right) dx$$

$$\underset{\frac{x+y}{2}\geq\sqrt{xy}}{\geq} 2\int_a^b \left(|(T^{q_n})'(x)|\cdot|(T^{-q_n})'(x)|\right)^{1/2}dx \geq 2C^{\frac{1}{2}}|I_0| \gg 1 \ \lightning$$

$\boxtimes$

For the next lemma, we used the notion of a cyclic order.

Definition

Let X a set. A subset $C\subseteq X^3$ is a cyclic order if

1. $\forall (a,b,c)\in C \quad (b,c,a)\in C$ (cyclicity),
2. $\forall (a,b,c)\in C \quad (c,b,a)\notin C$ (asymmetry),
3. $\forall a,b,c,d\in X \quad (a,b,c)\in C \wedge (a,c,d)\in C \Rightarrow (a,b,d)\in C$ (transitivity),
4. $\forall a,b,c\in X$ pairwise distinct $\quad (a,b,c)\in C \vee (c,b,a)\in C$ (connectedness).

Exercise:

The subset $C\subseteq \mathbb{T}^3$ given by

$$(x_0,x_1,x_2)\in C \iff \exists \varphi_0\in[0,1)\ \exists 0\leq t_0<t_1<t_2<1 \quad x_i=\varphi_0+t_i \bmod 1$$

is a cyclic order on $\mathbb{T}$.

In what follows, we denote the elements of C by $[x_0,x_1,x_2]$.

~~Lemma:~~

~~Let $x_0,x_1,x_2\in\mathbb{T}$, $\varrho\in\mathbb{R}$, then $[R_\varrho x_0, R_\varrho x_1, R_\varrho x_2]$ if and only if there exists a cyclic permutation $\sigma:\{0,1,2\}\to\{0,1,2\}$ and $m\in\mathbb{Z}$ s.t.~~

$$\cancel{\hat{R}_\varrho x_{\sigma(0)}+m < \hat{R}_\varrho x_{\sigma(1)}+m < \hat{R}_\varrho x_{\sigma(2)}+m}$$

~~Similarly for T.~~

~~Proof skipped/exercise.~~

$$\cancel{[R_\varrho x_0, R_\varrho x_1, R_\varrho x_2] = [x_0+\varrho, x_1+\varrho, x_2+\varrho] \iff \exists\sigma \text{ a cyclic permutation and } 0\leq t_0<t_1<t_2<1 \text{ s.t. } x_{\sigma(i)}+\varrho+m=t_i}$$

Lemma

Let $T\in \mathrm{Homeo}(\mathbb{T})$, $\varrho=\varrho(T)\notin \mathbb{Q}/\mathbb{Z}$. Then for every $x\in\mathbb{T}$ and for all $u_0,u_1,u_2\in\mathbb{Z}$

$$[T^{u_0}x, T^{u_1}x, T^{u_2}x] \iff [R_\varrho^{u_0}x, R_\varrho^{u_1}x, R_\varrho^{u_2}x]$$

Proof: $[T^{u_0}x, T^{u_1}x, T^{u_2}x] \iff \exists\sigma:\{0,1,2\}\to\{0,1,2\}$ a cyclic permutation and $m_0,m_1,m_2\in\mathbb{Z}$ s.t.

$$0\leq \hat{T}^{u_{\sigma(0)}}(x)+m_{\sigma(0)} < \hat{T}^{u_{\sigma(1)}}(x)+m_{\sigma(1)} < \hat{T}^{u_{\sigma(2)}}(x)+m_{\sigma(2)} < 1$$

$$\overset{\text{last time}}{\iff} \hat{T}^{u_{\sigma(2)}}(0)+m_{\sigma(2)}-1 < \hat{T}^{u_{\sigma(0)}}(0)+m_{\sigma(0)} < \hat{T}^{u_{\sigma(1)}}(0)+m_{\sigma(1)} < \hat{T}^{u_{\sigma(2)}}(0)+m_{\sigma(2)}$$

$$\overset{\text{last time}}{\iff} u_{\sigma(2)}\varrho+m_{\sigma(2)}-1 < u_{\sigma(0)}\varrho+m_{\sigma(0)} < u_{\sigma(1)}\varrho+m_{\sigma(1)} < u_{\sigma(2)}\varrho+m_{\sigma(2)}$$

$$\iff [R_\varrho^{u_{\sigma(0)}}(0), R_\varrho^{u_{\sigma(1)}}(0), R_\varrho^{u_{\sigma(2)}}(0)] \iff [R_\varrho^{u_0}(0), R_\varrho^{u_1}(0), R_\varrho^{u_2}(0)]$$

$$\iff [R_\varrho^{u_0}(x), R_\varrho^{u_1}(x), R_\varrho^{u_2}(x)].$$

$\boxtimes$

Lemma

Suppose $T\in\mathrm{Homeo}(\mathbb{T})\cap C^1(\mathbb{T})$ is orientation preserving, of bounded logarithmic total variation, and $\varrho(T)\notin\mathbb{Q}$. Then there exists $(q_n)_{n\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$ $q_n\nearrow\infty$ s.t.

$$\forall x\in\mathbb{T}\ \forall n\in\mathbb{N} \quad \{(T^k x, T^{k+q_n}x)+\mathbb{Z} : 0\leq k<q_n\}\subseteq 2^{\mathbb{T}}$$

are pairwise disjoint.

End of lecture 6

Proof: by the preceding lemma, it suffices to prove this for $T=R_\varrho$, where $\varrho=\varrho(T)$.

Define $(q_n)_{n\in\mathbb{N}}$ as follows.

- $q_1=1$
- $q_{n+1}=\min\{q\in\mathbb{N} : d(T^q x,x) < \underbrace{\min\{d(T^\ell x,x) : 1\leq\ell\leq q_n\}}\}$,

that is, q_n is the sequence of times of closest recurrences. This is independent of x, since $d(T^\ell x,x)=d(x+\varrho\ell,x)=d(y+\varrho\ell,y)=d(T^\ell y,y)$

Let $\delta_n := d(T^{q_n}x,x)<\frac{1}{2}$ (since ϱ is irrational)

Note: $\cdot\ \varrho\notin\mathbb{Q}/\mathbb{Z} \Rightarrow \alpha_n>0$,

$\cdot\ T$ minimal $\Rightarrow \mathcal{O}^+(x)$ dense $\Rightarrow \forall\varepsilon>0\ \exists q\in\mathbb{N}\ d(T^q x,x)<\varepsilon$.

Let now $0\leq k<q_n$ and $I:=(T^k x, T^{q_n+k}x)$. Write $x_i := T^i x$. Suppose for sake of contradiction that $\exists 0\leq r<2q_n$ such that $x_r\in I$. Note that $r\neq i$.

Case 1: If $r<k$, then $x_0=T^{-r}(x_r)\in T^{-r}(x_k,x_{q_n+k})=(x_{k-r},x_{q_n+k-r})$.

In particular

$$d(x_{k-r},x_0)<d(x_{k-r},x_{q_n+k-r})=d(x_0,x_{q_n})=\delta_n,$$

which is absurd since $0<k-r\leq q_n$.

Case 2: If $r>k$, then

$$x_{r-k}=T^{-k}(x_r)\in T^{-k}(x_k,x_{q_n+k})=(x_0,x_{q_n}).$$

Hence $d(x_{r-k},x_0)<d(x_{q_n},x_0)$ implies that $r-k>q_n$. So

$$x_{(r-k)-q_n}=T^{-q_n}(x_{r-k})\in T^{-q_n}(x_0,x_{q_n})=(x_{-q_n},x_0),$$

$$\therefore d(x_{r-k-q_n},x_0)<d(x_{-q_n},x_0)=d(x_0,x_{q_n})=\delta_n,$$

which is absurd since $r-k<2q_n$.

$\boxtimes$