$$\geq 2 \int_a^b \left(|(T^{q_n})'(x)| \cdot |(T^{-q_n})'(x)| \right)^{1/2} dx \geq 2 C^{\frac{1}{2}} |I_0| >> 1 \quad ⚡$$

$\uparrow$ $\frac{x+y}{2} \geq \sqrt{xy}$

$\boxtimes$

For the next lemma, we need the notion of a cyclic order.

Definition

Let X a set. A subset $C \subseteq X^3$ is a cyclic order if

1. $\forall (a,b,c) \in C \quad (b,c,a) \in C$ (cyclicity),
2. $\forall (a,b,c) \in C \quad (c,b,a) \notin C$ (asymmetry),
3. $\forall a,b,c,d \in X \quad (a,b,c) \in C \wedge (a,c,d) \in C \Rightarrow (a,b,d) \in C$ (transitivity),
4. $\forall a,b,c \in X$ pairwise distinct $\quad (a,b,c) \in C \vee (c,b,a) \in C$ (connectedness).

Exercise:

The subset $C \subseteq \mathbb{T}^3$ given by

$$(x_0, x_1, x_2) \in C \iff \exists \varphi_0 \in [0,1) \, \exists \, 0 \leq t_0 < t_1 < t_2 < 1 \quad x_i = \varphi_0 + t_i \mod 1$$

is a cyclic order on $\mathbb{T}$.

In what follows, we denote the elements of C by $[x_0, x_1, x_2]$.

~~Lemma:~~

~~Let $x_0, x_1, x_2 \in \mathbb{T}$, $\varrho \in \mathbb{R}$, then $[R_\varrho x_0, R_\varrho x_1, R_\varrho x_2]$ if and only if there exists a cyclic permutation $\sigma : \{0,1,2\} \to \{0,1,2\}$ and $m \in \mathbb{Z}$ s.t.~~

~~$$\hat{R}_\varrho x_{\sigma(0)} + m < \hat{R}_\varrho x_{\sigma(1)} + m < \hat{R}_\varrho x_{\sigma(2)} + m$$~~

~~Similarly for T.~~

~~Proof skipped/exercise.~~

~~$[R_\varrho x_0, R_\varrho x_1, R_\varrho x_2] = [x_0+\varrho, x_1+\varrho, x_2+\varrho] \iff \exists \sigma$ a cyclic permutation and $0 \leq t_0 < t_1 < t_2 < 1$ s.t. $x_{\sigma(i)} + \varrho + m = t_i$~~

Lemma

Let $T \in \mathrm{Homeo}(\mathbb{T})$, $\varrho = \varrho(T) \notin \mathbb{Q}/\mathbb{Z}$. Then for every $x \in \mathbb{T}$ and for all $u_0, u_1, u_2 \in \mathbb{Z}$

$$[T^{u_0}x, T^{u_1}x, T^{u_2}x] \iff [R_\varrho^{u_0}x, R_\varrho^{u_1}x, R_\varrho^{u_2}x]$$

Proof: $[T^{u_0}x, T^{u_1}x, T^{u_2}x] \iff \exists \sigma : \{0,1,2\} \to \{0,1,2\}$ a cyclic permutation and $m_0, m_1, m_2 \in \mathbb{Z}$ s.t.

$$0 \leq \hat{T}^{u_{\sigma(0)}}(x) + m_{\sigma(0)} < \hat{T}^{u_{\sigma(1)}}(x) + m_{\sigma(1)} < \hat{T}^{u_{\sigma(2)}}(x) + m_{\sigma(2)} < 1$$

$$\overset{\text{last time}}{\iff} \hat{T}^{u_{\sigma(2)}}(0) + m_{\sigma(2)} - 1 < \hat{T}^{u_{\sigma(0)}}(0) + m_{\sigma(0)} < \hat{T}^{u_{\sigma(1)}}(0) + m_{\sigma(1)} < \hat{T}^{u_{\sigma(2)}}(0) + m_{\sigma(2)}$$

$$\overset{\text{last time}}{\iff} u_{\sigma(2)}\varrho + m_{\sigma(2)} - 1 < u_{\sigma(0)}\varrho + m_{\sigma(0)} < u_{\sigma(1)}\varrho + m_{\sigma(1)} < u_{\sigma(2)}\varrho + m_{\sigma(2)}$$

$$\iff [R_\varrho^{u_{\sigma(0)}}(0), R_\varrho^{u_{\sigma(1)}}(0), R_\varrho^{u_{\sigma(2)}}(0)] \iff [R_\varrho^{u_0}(0), R_\varrho^{u_1}(0), R_\varrho^{u_2}(0)]$$

$$\iff [R_\varrho^{u_0}(x), R_\varrho^{u_1}(x), R_\varrho^{u_2}(x)].$$

$\boxtimes$

Lemma

Suppose $T \in \mathrm{Homeo}(\mathbb{T}) \cap C^1(\mathbb{T})$ is orientation preserving, of bounded logarithmic total variation, and $\varrho(T) \notin \mathbb{Q}$. Then there exists $(q_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ $q_n \nearrow \infty$ s.t.

$$\forall x \in \mathbb{T} \, \forall n \in \mathbb{N} \quad \{ (T^k x, T^{k+q_n} x) + \mathbb{Z} : 0 \leq k < q_n \} \subseteq 2^{\mathbb{T}}$$

are pairwise disjoint.

End of lecture 6

Proof: by the preceding lemma, it suffices to prove this for $T = R_\varrho$, where $\varrho = \varrho(T)$.

Define $(q_n)_{n \in \mathbb{N}}$ as follows.

- $q_1 = 1$
- $q_{n+1} = \min \{ q \in \mathbb{N} : d(T^q x, x) < \underline{\min \{ d(T^\ell x, x) : 1 \leq \ell \leq q_n \}} \}$,

that is, q_n is the sequence of times of closest recurrences. This is independent of x, since $d(T^\ell x, x) = d(x + \varrho \ell, x) = d(y + \varrho \ell, y) = d(T^\ell y, y)$

Let $\delta_n := d(T^{q_n} x, x) \leq \frac{1}{2}$ (since ϱ is irrational)

Note: $\cdot \, \varrho \notin \mathbb{Q}/\mathbb{Z} \Rightarrow \alpha_n > 0$,

$\cdot \, T$ minimal $\Rightarrow \mathcal{O}^+(x)$ dense $\Rightarrow \forall \varepsilon > 0 \, \exists q \in \mathbb{N} \; d(T^q x, x) < \varepsilon$.

Let now $0 \leq k < q_n$ and $I := (T^k x, T^{q_n + k} x)$. Write $x_i := T^i x$. Suppose for sake of contradiction that $\exists \, 0 \leq r < 2q_n$ such that $x_r \in I$. Note that $r \neq i$.

Case 1: If $r < k$, then $x_0 = T^{-r}(x_r) \in T^{-r}(x_k, x_{q_n+k}) = (x_{k-r}, x_{q_n+k-r})$.

In particular

$$d(x_{k-r}, x_0) < d(x_{k-r}, x_{q_n+k-r}) = d(x_0, x_{q_n}) = \delta_n,$$

which is absurd since $0 < k - r \leq q_n$.

Case 2: If $r > k$, then

$$x_{r-k} = T^{-k}(x_r) \in T^{-k}(x_k, x_{q_n+k}) = (x_0, x_{q_n}).$$

Hence $d(x_{r-k}, x_0) < d(x_{q_n}, x_0)$ implies that $r - k > q_n$. So

$$x_{(r-k)-q_n} = T^{-q_n}(x_{r-k}) \in T^{-q_n}(x_0, x_{q_n}) = (x_{-q_n}, x_0),$$

$\therefore d(x_{r-k-q_n}, x_0) < d(x_{-q_n}, x_0) = d(x_0, x_{q_n}) = \delta_n$,

which is absurd since $r - k < 2q_n$.

$\boxtimes$

Proof of the theorem: Recall that T has bounded logarithmic total variation and that we know the following: Suppose that there is $q_n \uparrow \infty$ such that for all $x \in \mathbb{T}$

$$|(T^{q_n})'(x)| \cdot |(T^{-q_n})'(x)| \leq \exp(-\mathrm{Var}(\log|T'|)), \qquad (*)$$

then T is minimal.

So let $q_n \uparrow \infty$ as in the preceding lemma, i.e., the intervals

$$\{T^k(x, T^{q_n}x) : 0 \leq k < q_n\} = \{T^k(x_0, x_{q_n}) : 0 \leq k < q_n\}$$

are disjoint. Then, for any $n \geq 1$

$$\mathrm{Var}(\log|T'|) \overset{\text{def}}{=} \sup \left\{ \sum_{i=0}^{m-1} \left| \log|T'(y_{i+1})| - \log|T'(y_i)| \right| : 0 = y_0 < \dots < y_m = 1 \right\}$$

$$\overset{\text{possibly reorder}}{\geq} \sum_{k=0}^{q_n-1} \left| \log|T'(x_k)| - \log|T'(x_{q_n+k})| \right|$$

$$\geq \left| \sum_{k=0}^{q_n-1} \log|T'(x_k)| - \log|T'(x_{q_n+k})| \right|$$

$$= \left| \sum_{k=0}^{q_n-1} \log \frac{|T'(x_k)|}{|T'(x_{q_n+k})|} \right|$$

$$= \left| \log \frac{\prod_{k=0}^{q_n-1} |T'(T^k x)|}{\prod_{k=0}^{q_n-1} |T'(T^{q_n+k} x)|} \right|$$

$$= \left| \log \frac{\left|\prod_{k=0}^{q_n-1} T'(T^k x)\right|}{\left|\prod_{k=0}^{q_n-1} T'(T^k x_{q_n})\right|} \right|$$

$$= \left| \log \frac{|(T^{q_n})'(x_0)|}{|(T^{q_n})'(T^{q_n} x_0)|} \right|.$$

Recall that $(T^{-k})'(x) = \dfrac{1}{(T^k)'(T^k x)}$ $\quad \left[f \circ g = id \Rightarrow (f\circ g)'(x) = f'(g(x))g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))} \right.$

Hence $\exp(-\mathrm{Var}(\log|T'|)) \leq |(T^{q_n})'(x)| \, |(T^{-q_n})'(x)|$ as desired.

Remark/exercise: Bounded logarithmic variation is necessary.

End of lecture 7

II – Symbolic dynamics

In this section we'll discuss more examples related to the left-shift. This setting is surprisingly relevant, for example it is a model for the geodesic flow on certain surfaces (of interest).

Recall the definition of the full shift. Given an alphabet $A = \{0, \dots, p-1\}$, let

$$\begin{aligned} \sigma : A^{\mathbb{Z}} &\longrightarrow A^{\mathbb{Z}} \\ (x_n)_n &\longmapsto (x_{n+1})_n . \end{aligned}$$

We have seen in the exercises that $A^{\mathbb{Z}}$ is a compact metric space and that the cylinder sets

$${}_m[a_0, \dots, a_\ell] = \{(x_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}}, \forall 0 \leq k \leq \ell \;\; x_{m+k} = a_k\}$$

are open. Note that they are also closed, hence compact, as follows from

$$B_{2^{-m}}((x_n)_n) = {}_{-m}[x_{-m}, \dots, x_m] = \{y : d(y,x) \leq 2^{-m}\}.$$

Similar results hold for $A^{\mathbb{N}}$ equipped with the one-sided shift.

Definition (Subshifts)

Let $A = \{0, \dots, p-1\}$.

(i) A subshift of $A^{\mathbb{N}}$ is a non-empty closed subset $X \subseteq A^{\mathbb{N}}$ which is invariant, i.e., $\sigma X \subseteq X$, with associated dynamical system $(X, \sigma|_X) = (X, \sigma)$.

(ii) A subshift of $A^{\mathbb{Z}}$ is a non-empty closed subset $X \subseteq A^{\mathbb{Z}}$ which is strongly invariant, i.e., $\sigma X = X$, with associated dynamical system $(X, \sigma|_X) = (X, \sigma)$.

Examples

1) (Vertex shift) Let $\mathcal{G} = (V, E)$ be a finite graph, i.e., V is a finite set and $E \subseteq V \times V$. We let

$$X_{\mathcal{G}} = \{(x_n)_{n\in\mathbb{Z}} \in V^{\mathbb{Z}} : x \text{ describes the itinerary of a bi-infinite path on } \mathcal{G}\}$$
$$= \{(x_n)_{n\in\mathbb{Z}} \in V^{\mathbb{Z}} : \forall n \in \mathbb{Z} \;\; (x_n, x_{n+1}) \in E\}$$

$$X^+_{\mathcal{G}} = \{(x_n)_{n\in\mathbb{N}} \in V^{\mathbb{N}} : \forall n \in \mathbb{N} \;\; (x_n, x_{n+1}) \in E\}.$$

Identifying $V \longleftrightarrow \{0, \dots, |V|-1\}$ and noting that, e.g.,

$$X_{\mathcal{G}} = \bigcap_{n\in\mathbb{Z}} \bigcup_{e\in E} {}_n[e]$$