Proof of the theorem: Recall that T has bounded logarithmic total variation and that we know the following: Suppose that there is $q_n \uparrow \infty$ such that for all $x \in \mathbb{T}$

$$|(T^{q_n})'(x)| \cdot |(T^{-q_n})'(x)| \leq \exp(-\mathrm{Var}(\log|T'|)), \qquad (*)$$

then T is minimal.

So let $q_n \uparrow \infty$ as in the preceding lemma, i.e., the intervals

$$\{T^k(x, T^{q_n}x) : 0 \leq k < q_n\} = \{T^k(x_0, x_{q_n}) : 0 \leq k < q_n\}$$

are disjoint. Then, for any $n \geq 1$

$$\mathrm{Var}(\log|T'|) \overset{\text{def}}{=} \sup\left\{\sum_{i=0}^{m-1} \left|\log|T'(y_{i+1})| - \log|T'(y_i)|\right| : 0 = y_0 < \dots < y_m = 1\right\}$$

$$\overset{\text{possibly reorder}}{\geq} \sum_{k=0}^{q_n-1} \left|\log|T'(x_k)| - \log|T'(x_{q_n+k})|\right|$$

$$\geq \left|\sum_{k=0}^{q_n-1} \log|T'(x_k)| - \log|T'(x_{q_n+k})|\right|$$

$$= \left|\sum_{k=0}^{q_n-1} \log\frac{|T'(x_k)|}{|T'(x_{q_n+k})|}\right|$$

$$= \left|\log\frac{\prod_{k=0}^{q_n-1}|T'(T^k x)|}{\prod_{k=0}^{q_n-1}|T'(T^{q_n+k}x)|}\right|$$

$$= \left|\log\frac{|\prod_{k=0}^{q_n-1} T'(T^k x)|}{|\prod_{k=0}^{q_n-1} T'(T^k x_{q_n})|}\right|$$

$$= \left|\log\frac{|(T^{q_n})'(x_0)|}{|(T^{q_n})'(T^{q_n}x_0)|}\right|.$$

Recall that $(T^{-k})'(x) = \dfrac{1}{(T^k)'(T^k x)}$ $\quad$ [$f \circ g = id \Rightarrow (f\circ g)'(x) = f'(g(x))g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$]

Hence $\exp(-\mathrm{Var}(\log|T'|)) \leq |(T^{q_n})'(x)||(T^{-q_n})'(x)|$ as desired.

Remark/exercise: Bounded logarithmic variation is necessary. $\qquad$ End of lecture 7

II – Symbolic dynamics

In this section we'll discuss more examples related to the left-shift. This setting is surprisingly relevant, for example it is a model for the geodesic flow on certain surfaces (of interest).

Recall the definition of the full shift. Given an alphabet $A = \{0, \dots, p-1\}$, let

$$\sigma : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$$
$$(x_n)_n \longmapsto (x_{n+1})_n.$$

We have seen in the exercises that $A^{\mathbb{Z}}$ is a compact metric space and that the cylinder sets

$$_m[a_0, \dots, a_\ell] = \{(x_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}}, \forall 0 \leq k \leq \ell \quad x_{m+k} = a_k\}$$

are open. Note that they are also closed, hence compact, as follows from

$$B_{2^{-m}}((x_n)_n) = {}_{-m}[x_{-m}, \dots, x_m] = \{y : d(y,x) \leq 2^{-m}\}.$$

Similar results hold for $A^{\mathbb{N}}$ equipped with the one-sided shift.

Definition (Subshifts)

Let $A = \{0, \dots, p-1\}$.

(i) A subshift of $A^{\mathbb{N}}$ is a non-empty closed subset $X \subseteq A^{\mathbb{N}}$ which is invariant, i.e., $\sigma X \subseteq X$, with associated dynamical system $(X, \sigma|_X) = (X, \sigma)$.

(ii) A subshift of $A^{\mathbb{Z}}$ is a non-empty closed subset $X \subseteq A^{\mathbb{Z}}$ which is strongly invariant, i.e., $\sigma X = X$, with associated dynamical system $(X, \sigma|_X) = (X, \sigma)$.

Examples

1) (Vertex shift) Let $\mathcal{G} = (V, E)$ be a finite graph, i.e., V is a finite set and $E \subseteq V \times V$. We let

$$X_{\mathcal{G}} = \{(x_n)_{n\in\mathbb{Z}} \in V^{\mathbb{Z}} : x \text{ describes the itinerary of a bi-infinite path on } \mathcal{G}\}$$
$$= \{(x_n)_{n\in\mathbb{Z}} \in V^{\mathbb{Z}} : \forall n \in \mathbb{Z} \ (x_n, x_{n+1}) \in E\}$$

$$X^+_{\mathcal{G}} = \{(x_n)_{n\in\mathbb{N}} \in V^{\mathbb{N}} : \forall n \in \mathbb{N} \ (x_n, x_{n+1}) \in E\}.$$

Identifying $V \longleftrightarrow \{0, \dots, |V|-1\}$ and noting that, e.g.,

$$X_{\mathcal{G}} = \bigcap_{n\in\mathbb{Z}} \bigcup_{e\in E} {}_n[e]$$

one sees that $X_{\mathcal{G}}$ is closed and σ-invariant.

Note $X_{\mathcal{G}}$ could be empty (and similarly for $X_{\mathcal{G}}^+$), e.g.,

(in class exercise) 

$$\mathcal{G} = (\{0,1,2\}, \{(0,1),(1,2)\})$$

Lemma:

Let $\mathcal{G} = (V,E)$ a finite graph, then

$$X_{\mathcal{G}}^+, X_{\mathcal{G}} \neq \emptyset \iff \mathcal{G} \text{ has a closed loop.}$$

Proof: "$\Longleftarrow$" clear

"$\Longrightarrow$" Let $x \in X_{\mathcal{G}}$. Since V is finite, there are $n \in \mathbb{Z}$, $k \in \mathbb{N}$, such that

$$x_n = x_{n+k}.$$

Then $\{(x_n, x_{n+1}), \dots, (x_{n+k-1}, x_n)\}$ is a closed loop.

One argues similarly for $X_{\mathcal{G}}^+$.

From now on, we will always assume that $X_{\mathcal{G}} \neq \emptyset$. $\boxtimes$

2) (A so-called shift of finite type) Let $A = \{0,1,2\}$, $X = \{(x_n)_n \in A^{\mathbb{N}} : \forall n \in \mathbb{N}\ x_n \neq 1\}$.

Clearly $\sigma X = X$. Moreover, we have seen that a sequence in $(A^{\mathbb{N}})^{\mathbb{N}}$ converges if and only if each coordinate eventually stabilizes. Hence $X \subseteq A^{\mathbb{N}}$ is closed.

From now on, we will focus on two-sided shifts, i.e., invariant closed subsets of $A^{\mathbb{Z}}$. Reasons for that will become apparent when we discuss factor maps.

Proposition (Minimal shifts are rather boring)

Let $\mathcal{G}$ be a finite graph, $X_{\mathcal{G}}$ non-empty. If $X_{\mathcal{G}}$ is minimal, then $X_{\mathcal{G}}$ consists of only one periodic orbit.

Proof: Let $(a_0, \dots, a_k) \equiv ((a_0,a_1), \dots, (a_{k-1}, a_0)) \in E^k$ be a closed loop in $\mathcal{G}$. Then the sequence defined by

$$\forall p \in \mathbb{Z}\ \forall 0 \leq q < k \quad x_{pk+q} = a_q$$

is an element of $X_{\mathcal{G}}$. If $(X_{\mathcal{G}}, \sigma)$ is minimal, $(x_n)_{n\in\mathbb{Z}}$ has dense orbit. Let

$y \in X_{\mathcal{G}}$ arbitrary and let $\varepsilon > 0$. Then $\exists m \in \mathbb{N}$ s.t.

$$d(\sigma^m(x), y) < \varepsilon$$

$\Rightarrow \forall N \in \mathbb{N}\ \exists m \in \mathbb{N}$ s.t.

$$[y_{-N}, \dots, y_N] = [\sigma^m(x)_{-N}, \dots, \sigma^m(x)_N]$$

Since x is periodic, this shows

$$\exists m \in \mathbb{N}\ \forall N \in \mathbb{N} \quad [y_{-N}, \dots, y_N] = [\sigma^m(x)_{-N}, \dots, \sigma^m(x)_N].$$

Hence $y = \sigma^m(x) \in \mathcal{O}(x)$, which is finite. $\boxtimes$

Given a graph $\mathcal{G}$ and a bijection $V \longleftrightarrow \{0, \dots, |V|-1\} = \{0, \dots, p-1\}$, define the <u>adjacency matrix</u>

$$A_{\mathcal{G}} = (a_{ij})_{i,j=0}^{p-1} \in \mathrm{Mat}_p(\mathbb{Z})$$

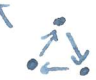
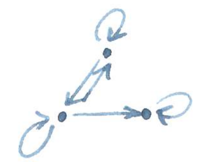
by

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{else.} \end{cases}$$

Remark: The map $\mathcal{G} \longmapsto A_{\mathcal{G}}$ is finite to finite, since $A_{\mathcal{G}}$ depends on the choice of the identification $V \cong \{0, \dots, p-1\}$.

Every element in $\bigcup_{p \geq 1} \mathrm{Mat}_p(\{0,1\})$ gives rise to a finite graph.

<u>In class exercise:</u> Determine

- A  $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
- A  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
- $\mathcal{G}$ for $A_{\mathcal{G}} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ 
- $\mathcal{G}$ for $A_{\mathcal{G}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ 

Lemma Let $\mathcal{G}=(V,E)$ a finite graph and $|V|=p$, $V\cong\{0,\dots,p-1\}$

Let $n\in\mathbb{N}$, $0\le i,j<p$. The following are equivalent:

(i) $\exists\ \underbrace{(a_0,\dots,a_n)\in V^{n+1} \text{ s.t. } (a_0,a_1),\dots,(a_{n-1},a_n)\in E \text{ and } a_0=i, a_n=j}_{(a_0,\dots,a_n) \text{ is admissible}}$.

(ii) $(A_{\mathcal{G}}^n)_{ij}\neq 0$.

Moreover, $(A_{\mathcal{G}}^n)_{ij}$ is exactly the number of paths of length n from i to j.

Proof: If $n=1$, this is the definition of $A_{\mathcal{G}}$. We proceed by induction.

Let $n\in\mathbb{N}$ and suppose the conclusion of the lemma is true for n and for all $0\le i,j<p$. Let $0\le i,j<p$ arbitrary, then

$$(A_{\mathcal{G}}^{n+1})_{ij}=\sum_{k=0}^{p-1}(A_{\mathcal{G}}^n)_{ik}\cdot(A_{\mathcal{G}})_{kj}.$$

By induction, $(A_{\mathcal{G}}^n)_{ik}$ is exactly the number of paths of length n from i to k.

By definition, $(A_{\mathcal{G}})_{kj}=\begin{cases}1 & \text{if } (k,j)\in E,\\ 0 & \text{else.}\end{cases}$

Hence $(A_{\mathcal{G}}^n)_{ik}\cdot(A_{\mathcal{G}})_{kj}$ is exactly the number of paths $(a_0,\dots,a_{n+1})$ from $a_0=i$ to $a_{n+1}=j$ such that $a_n=k$. Summing over k yields the number of paths of length $n+1$ from i to j. $\boxtimes$

Definition

Let $\mathcal{G}=(V,E)$ be a finite graph with adjacency matrix $A_{\mathcal{G}}$. We call $A_{\mathcal{G}}$

(i) <u>irreducible</u> if
$$\forall i,j\ \exists n\in\mathbb{N}\ (A_{\mathcal{G}}^n)_{ij}\neq 0;$$

(ii) <u>aperiodic</u> if
$$\exists n\in\mathbb{N}\ \forall i,j\ (A_{\mathcal{G}}^n)_{ij}\neq 0.$$

Remark: I will/might sometimes call the graph $\mathcal{G}$ irreducible/aperiodic meaning that $A_{\mathcal{G}}$ has the corresponding property. This might be non-standard.

Note: $A_{\mathcal{G}}$ aperiodic $\Rightarrow \exists n_0\in\mathbb{N}\ \forall n\ge n_0\ \forall i,j\ (A_{\mathcal{G}}^n)_{ij}\neq 0$.

Proof: Suppose that $A_{\mathcal{G}}$ has a zero column, then $A_{\mathcal{G}}^n$ has the same zero column for all $n\in\mathbb{N}$. Since $A_{\mathcal{G}}$ is aperiodic, this shows that $A_{\mathcal{G}}$ has no zero column.

Now suppose that $n\in\mathbb{N}$ and $A_{\mathcal{G}}^n$ has no zero entries. Since $A_{\mathcal{G}}\in \mathrm{Mat}_{|V|}(\{0,1\})$, all entries of $A_{\mathcal{G}}^n$ are strictly positive. Hence, for all $0\le i,j<|V|$, we have

$$(A_{\mathcal{G}}^{n+1})_{ij}=\sum_{k=0}^{|V|-1}\underbrace{(A_{\mathcal{G}}^n)_{ik}}_{>0}\cdot\underset{\substack{\uparrow\\ \text{not all simultaneously}\\ \text{zero}}}{(A_{\mathcal{G}})_{kj}}>0.$$

Proposition

Let $\mathcal{G}=(V,E)$ be a finite graph with adjacency matrix $A_{\mathcal{G}}$ and suppose $X_{\mathcal{G}}\neq\emptyset$.

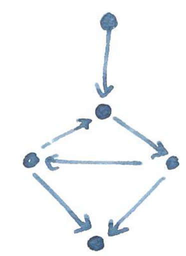
Then

(i) $A_{\mathcal{G}}$ irreducible $\Rightarrow (X_{\mathcal{G}},\sigma)$ is (forward) transitive, and

(ii) $A_{\mathcal{G}}$ periodic $\Rightarrow (X_{\mathcal{G}},\sigma)$ is topologically mixing.

Moreover, in either case the periodic points are dense.

In class exercise: Find $\mathcal{G}$ such that $(X_{\mathcal{G}}^+,\sigma)$ is transitive but $A_{\mathcal{G}}$ not irreducible and one such that $(X_{\mathcal{G}}^+,\sigma)$ is mixing but $A_{\mathcal{G}}$ not aperiodic.



(sinks and sources)

Proof: (i) Suppose $A_{\mathcal{G}}$ is irreducible. We want to prove transitivity and, hence, need to show that

$$\forall U,V\subseteq X_{\mathcal{G}} \text{ open non-empty } \exists r\in\mathbb{Z} \text{ s.t. } \sigma^r(U)\cap V\neq\emptyset.$$

We can w.l.o.g. assume that $U={}_m[a_0,\dots,a_\ell]$ and $V={}_M[A_0,\dots,A_L]$. Since $A_{\mathcal{G}}$ is irreducible, there exists $n\in\mathbb{N}$ s.t. $(A_{\mathcal{G}}^n)_{a_\ell,A_0}\neq 0$.

$\therefore \exists x_0,\dots,x_n \in V$ s.t. $x_0 = a_\ell, x_n = A_0$, and
$\forall i = 1,\dots,n \quad (x_{i-1}, x_i) \in E$.

In particular, $\emptyset \neq {}_m[a_0,\dots,a_\ell,x_1,\dots,x_n,A_1,\dots,A_L] \subseteq U$ and, letting $k = m-M+\ell+n$ we have

$$\emptyset \neq \sigma^k\left({}_m[a_0,\dots,a_\ell,x_1,\dots,x_n,A_1,\dots,A_L]\right)$$

$$= {}_{m-k}[a_0,\dots,a_\ell,x_1,\dots,x_n,A_1,\dots,A_L]$$

$$= {}_{M-\ell-n}[a_0,\dots,a_\ell,x_1,\dots,x_n,A_1,\dots,A_L]$$

$$\subseteq {}_M[x_n,A_1,\dots,A_L] \subseteq V.$$

For density of periodic points, let $U \subseteq X_{\mathcal{G}}$ open, ${}_m[a_0,\dots,a_\ell] \subseteq U$. By irreducibility of $A_{\mathcal{G}}$, there exists $(b_0,\dots,b_r) \in V^{r+1}$ admissible such that $b_0 = a_\ell$, $b_r = a_0$. Define

$$\forall p \in \mathbb{Z}\ \forall 0 \leq q < \ell+r \quad x^*_{p(\ell+r)+q} = \begin{cases} a_q & \text{if } 0 \leq q \leq \ell, \\ b_{q-(\ell+1)} & \text{else.} \end{cases}$$

Then $\sigma^m(x^*)$ is σ-periodic and $\sigma^m(x^*) \in U$.

(ii) Suppose $A_{\mathcal{G}}$ is aperiodic. Let $U, V \subseteq X_{\mathcal{G}}$ open and ${}_m[a_0,\dots,a_\ell] \subseteq U$, ${}_M[A_0,\dots,A_L] \subseteq V$.

Given $r \in \mathbb{N}$, we have $\sigma^r({}_m[a_0,\dots,a_\ell]) = {}_{m-r}[a_0,\dots,a_\ell]$. Let $n_0 \in \mathbb{N}$ s.t.

$$\forall i,j\ \forall n \geq n_0 \quad (A^n_{\mathcal{G}})_{i,j} > 0.$$

Let $n_1 = m - M + \ell + n_0$.

Claim:

$\forall n > n_1 \quad \sigma^n({}_m[a_0,\dots,a_\ell]) \cap {}_M[A_0,\dots,A_L] \neq \emptyset$

Note that

$$x \in \sigma^n({}_m[a_0,\dots,a_\ell]) \cap {}_M[A_0,\dots,A_L] \Leftrightarrow x_{m-n} = a_0,\dots,x_{m-n+\ell} = a_\ell, x_M = A_0,\dots,x_{M+L} = A_L.$$

Since

$$M-(m-n+\ell) = M-m+n-\ell > n_0,$$

we know that

$$\left(A^{M-(m-n+\ell)}_{\mathcal{G}}\right)_{a_\ell, A_0} > 0,$$

Thus there exists an admissible word connecting the conditions, i.e., $\exists y \in V^{M-m+n-\ell+1}$ s.t. $y_0 = a_\ell,\dots,y_{M-m+n-\ell} = A_0$, hence

$$\emptyset \neq {}_{m-n}[a_0,\dots,a_\ell,y_1,\dots,y_{M-m+n-\ell},A_1,\dots,A_L] \subseteq \sigma^n({}_m[a_0,\dots,a_\ell]) \cap {}_M[A_0,\dots,A_L].$$

$\blacksquare$

Remark: A similar argument shows that $(X^+_{\mathcal{G}}, \sigma)$ is exact, i.e., if $A_{\mathcal{G}}$ is aperiodic, then $\forall U \subseteq X^+_{\mathcal{G}}$ open non-empty $\exists n \in \mathbb{N}$ s.t. $\sigma^n(U) = X_{\mathcal{G}}$.

Lemma:

Let $\mathcal{G} = (V,E)$ a finite graph, $X_{\mathcal{G}} \neq \emptyset$. Then
$$\forall n \in \mathbb{N} \quad tr(A^n_{\mathcal{G}}) = |\mathrm{Fix}(X_{\mathcal{G}}, \sigma^n)|$$

End of lecture 8

Proof: Immediate.

Definition (Shift of finite type)

A shift of finite type (abbr. sft) is a closed shift-invariant subset $X \subseteq A^{\mathbb{Z}}$ defined by a finite list of forbidden words, i.e.,

$$\exists L \in \mathbb{N}\ \exists F \subsetneq A^{L+1} \quad X = \{x \in A^{\mathbb{Z}} : \forall n \in \mathbb{Z} \ (x_n,\dots,x_{n+L}) \notin F\}$$

Proposition

Every shift of finite type is conjugate to a vertex shift.

Proof: Let X be a sft and $F \subsetneq A^{L+1}$ the set of forbidden words. Let $V := A^L$ and define

$$E = \{(v,v') \in V^2 : \exists a_0, a_{L+1} \in A \text{ s.t. } va_{L+1} = a_0v' \in A^{L+1} - F\}.$$

Define $h: X \longrightarrow X_{(V,E)}$ by

$$(x_n)_{n\in\mathbb{Z}} \longmapsto ((x_n,\dots,x_{n+L-1}))_{n\in\mathbb{Z}}.$$

Then h is continuous with continuous inverse and by construction

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[d, "h"] & X \arrow[d, "h"] \\ X_{(V,E)} \arrow[r, "\sigma_{(V,E)}"] & X_{(V,E)} \end{tikzcd}$$

$\blacksquare$