$\therefore \exists x_0,\dots,x_n \in V$ s.t. $x_0 = a_\ell, x_n = A_0$, and

$$\forall i = 1,\dots,n \qquad (x_{i-1}, x_i) \in E.$$

In particular, $\emptyset \neq {}_m[a_0,\dots,a_\ell,x_1,\dots,x_n,A_1,\dots,A_L] \subseteq U$ and, letting $k = m-M+\ell+n$ we have

$$\emptyset \neq \sigma^k\left({}_m[a_0,\dots,a_\ell,x_1,\dots,x_n,A_1,\dots,A_L]\right)$$

$$= {}_{m-k}[a_0,\dots,a_\ell,x_1,\dots,x_n,A_1,\dots,A_L]$$

$$= {}_{M-\ell-n}[a_0,\dots,a_\ell,x_1,\dots,x_n,A_1,\dots,A_L]$$

$$\subseteq {}_M[x_n,A_1,\dots,A_L] \subseteq V.$$

For density of periodic points, let $U \subseteq X_{\mathcal{G}}$ open, ${}_m[a_0,\dots,a_\ell] \subseteq U$. By irreducibility of $A_{\mathcal{G}}$, there exists $(b_0,\dots,b_r) \in V^{r+1}$ admissible such that $b_0 = a_\ell, b_r = a_0$. Define

$$\forall p \in \mathbb{Z}\ \forall 0 \leq q < \ell+r \qquad x^*_{p(\ell+r)+q} = \begin{cases} a_q & \text{if } 0 \leq q \leq \ell, \\ b_{q-(\ell+1)} & \text{else.} \end{cases}$$

Then $\sigma^m(x^*)$ is σ-periodic and $\sigma^m(x^*) \in U$.

(ii) Suppose $A_{\mathcal{G}}$ is aperiodic. Let $U, V \subseteq X_{\mathcal{G}}$ open and ${}_m[a_0,\dots,a_\ell] \subseteq U$, ${}_M[A_0,\dots,A_L] \subseteq V$.

Given $r \in \mathbb{N}$, we have $\sigma^r({}_m[a_0,\dots,a_\ell]) = {}_{m-r}[a_0,\dots,a_\ell]$. Let $n_0 \in \mathbb{N}$ s.t.

$$\forall i,j\ \forall n \geq n_0 \quad (A_{\mathcal{G}}^n)_{i,j} > 0.$$

Let $n_1 = m - M + \ell + n_0$.

Claim:

$$\forall n > n_1 \quad \sigma^n({}_m[a_0,\dots,a_\ell]) \cap {}_M[A_0,\dots,A_L] \neq \emptyset$$

Note that

$$x \in \sigma^n({}_m[a_0,\dots,a_\ell]) \cap {}_M[A_0,\dots,A_L] \Leftrightarrow x_{m-n} = a_0,\dots,x_{m-n+\ell} = a_\ell, x_M = A_0,\dots,x_{M+L} = A_L.$$

Since

$$M - (m-n+\ell) = M - m + n - \ell > n_0,$$

we know that

$$\left(A_{\mathcal{G}}^{M-(m-n+\ell)}\right)_{a_\ell, A_0} > 0,$$

Thus there exists an admissible word connecting the conditions, i.e., $\exists y \in V^{M-m+n-\ell+1}$ s.t. $y_0 = a_\ell, \dots, y_{M-m+n-\ell} = A_0$, hence

$$\emptyset \neq {}_{m-n}[a_0,\dots,a_\ell,y_1,\dots,y_{M-m+n-\ell},A_1,\dots,A_L] \subseteq \sigma^n({}_m[a_0,\dots,a_\ell]) \cap {}_M[A_0,\dots,A_L].$$

▨

Remark: A similar argument shows that $(X_{\mathcal{G}}^+, \sigma)$ is exact, i.e., if A_G is aperiodic, then $\forall U \subseteq X_{\mathcal{G}}^+$ open non-empty $\exists n \in \mathbb{N}$ s.t. $\sigma^n(U) = X_{\mathcal{G}}$.

Lemma:

Let $\mathcal{G} = (V,E)$ a finite graph, $X_{\mathcal{G}} \neq \emptyset$. Then

$$\forall n \in \mathbb{N} \quad \operatorname{tr}(A_{\mathcal{G}}^n) = |\operatorname{Fix}(X_{\mathcal{G}}, \sigma^n)|$$

End of lecture 8

Proof: Immediate.

Definition (Shift of finite type)

A shift of finite type (abbr. sft) is a closed shift-invariant subset $X \subseteq A^{\mathbb{Z}}$ defined by a finite list of forbidden words, i.e.,

$$\exists L \in \mathbb{N}\ \exists \mathcal{F} \subsetneq A^{L+1} \quad X = \{x \in A^{\mathbb{Z}} : \forall n \in \mathbb{Z} \ (x_n,\dots,x_{n+L}) \notin \mathcal{F}\}$$

Proposition

Every shift of finite type is conjugate to a vertex shift.

Proof: Let X be a sft and $\mathcal{F} \subsetneq A^{L+1}$ the set of forbidden words. Let $V := A^L$ and define

$$E = \{(v,v') \in V^2 : \exists a_0, a_{L+1} \in A \text{ s.t. } va_{L+1} = a_0 v' \in A^{L+1} - \mathcal{F}\}.$$

Define $h : X \longrightarrow X_{(V,E)}$ by

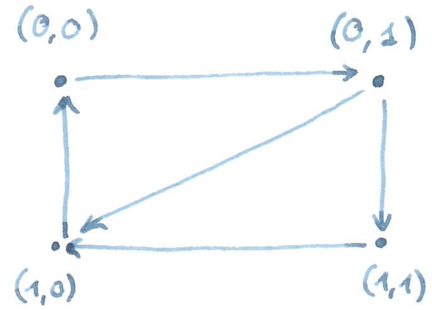
$$(x_n)_{n\in\mathbb{Z}} \longmapsto ((x_n,\dots,x_{n+L-1}))_{n\in\mathbb{Z}}.$$

Then h is continuous with continuous inverse and by construction

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ X_{(V,E)} \arrow[r, "\sigma_{(V,E)}"'] & X_{(V,E)} \end{tikzcd}$$

▨

__Example:__ $A=\{0,1\}$, $\mathcal{F}=\{(0,0,0),(1,1,1)\}$, then $\mathcal{G}=(V,E)$ becomes



Exercise (in class)

Step 1: find V

Step 2: draw all E

__Definition__

A __sofic shift__ is a shift-invariant closed subset $X\subseteq A^{\mathbb{Z}}$ s.t. there exists a shift of finite type $Y\subseteq B^{\mathbb{Z}}$ s.t. (Y,σ_Y) and (X,σ_X) are semiconjugate ((X,σ_X) is a topological factor of (Y,σ_Y), i.e., $\exists h: Y\longrightarrow X$ continuous and surjective such that

$$\begin{tikzcd} Y \arrow[r, "\sigma_Y"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "h"] \\ X \arrow[r, "\sigma_X"'] & X \end{tikzcd}$$

__Example:__

Let $X_{even}=\{x\in\{0,1\}^{\mathbb{Z}}$: any two 1s in x are separated by an even number of zeroes$\}$, e.g. $x=\ldots 100110000 1\ldots$, then X_{even} is a sofic shift (exercise).

__Hint:__ Construct a graph on the set $\{0', 0'', 1\}$.

and forget the primes.

Note that X_{even} is __not__ a shift of finite type since no finite set of words captures all the forbidden words $101, 10001, 1000001, \ldots$

Shifts of finite type can be very boring, e.g., $A=\{0,1\}$, $\mathcal{F}=\{(0,0),(1,0)\}$, then the associated shift of finite type has cardinality two.

The example $\mathcal{F}=\{(0,0,0),(1,1,1)\}$ is still rather restrictive but much much richer.

Can we quantify this?

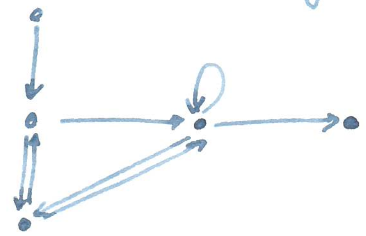
__Definition (Complexity)__

Let $X\subseteq A^{\mathbb{Z}}$ be a subshift. The complexity function for X is given by $p_X:\mathbb{N}\to\mathbb{N}$, mapping $n\in\mathbb{N}\longmapsto |\{\text{distinct words of length } n \text{ appearing in } X\}|$
$=|\pi_{\{0,\ldots,n-1\}}(X)|$ (by shift-invariance)

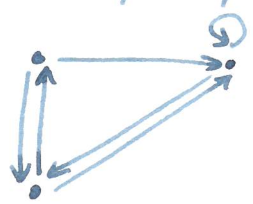
__Lemma:__

Let X be a shift of finite type. Then $p_X(n)$ grows either polynomially (at most) or it grows exponentially.

__Proof:__ As we have seen, we can assume that X is a vertex shift with graph $\mathcal{G}=(V,E)$ and adjacency matrix $A_{\mathcal{G}}$, e.g.,



We remove all sinks and sources to guarantee that any finite path can be extended to a bi-infinite path: $\forall v\in V\ \exists w,w'\in V\ (w,v),(v,w')\in E$:



It follows that $p_X(n)=\sum_{i,j} A^n_{\mathcal{G},ij}$. Note that this implies that

$$\|A^n_{\mathcal{G}}\|_\infty \le p_X(n)\le |V|^2\|A^n_{\mathcal{G}}\|_\infty.$$

Computing the Jordan normal form $A_{\mathcal{G}}=GJG^{-1}$, it follows that there is some $C\ge 1$, namely $C=|V|^4\|G\|_\infty\|G^{-1}\|_\infty$, such that

$$\forall n\in\mathbb{N}\quad \frac{1}{C}\|J^n\|_\infty\le\|A^n\|_\infty\le C\|J^n\|_\infty.$$

Now $\|J^n\|_\infty$ grows either polynomially or exponentially, e.g.,

$$\begin{pmatrix}\lambda & 1\\ & \lambda\end{pmatrix}^n=\begin{pmatrix}\lambda^n & n\lambda^{n-1}\\ & \lambda^n\end{pmatrix}.$$

__Theorem (Morse–Hedlund)__

Let X be a shift space. Then $|X|<\infty \iff \exists n\in\mathbb{N}\ p_X(n)\le n$.

__Proof:__ "$\Rightarrow$" Suppose $|X|<\infty$. Since $p_X(n)\le|X|$, the claim follows.

"$\Leftarrow$" Suppose $p_X(n)\le n$ and note that p_X is monotonic, i.e.,

$$\forall k\in\mathbb{N}\quad p_X(k)\le p_X(k+1).$$

Hence there exists a minimal such n and we have $p_X(n)=n$. Indeed, by minimality we have $n-1<p_X(n-1)\le p_X(n)\le n$ (since $p_X(1)\ge 1$).

If $n=1$, shift invariance implies that only one letter from the alphabet is being used, hence X is a singleton.

So suppose that $n>1$. By minimality, we have that $p_X(n-1)=p_X(n)=n$. Let $\mathcal{L}_n=\{w_1,\dots,w_n\}\subseteq A^n$ be the words of length n appearing in X. Let

$$L: \mathcal{L}_n \longrightarrow \mathcal{L}_{n-1}$$
$$w_i=(a_0,\dots,a_{n-1}) \longmapsto (a_1,\dots,a_{n-1}),$$
$$R: \mathcal{L}_n \longrightarrow \mathcal{L}_{n-1}$$
$$w_i=(a_0,\dots,a_{n-1}) \longmapsto (a_0,\dots,a_{n-2}).$$

Claim:
| L, R are bijections

We only prove it for L.

Surjectivity: Let $v\in\mathcal{L}_{n-1}$ and let $x\in X$ s.t. v appears in x. By ~~shift~~ shift-invariance we can assume w.l.o.g. that $x_1=v_1,\dots,x_{n-1}=v_{n-1}$.
Then $(x_0,v_1,v_2,\dots,v_{n-1})\in\mathcal{L}_n$ and this is mapped to v under L.

Injectivity: Since $p_X(n)=|\mathcal{L}_n|=|\mathcal{L}_{n-1}|$ as argued above, surjectivity implies that L is a bijection.

Using the claim, we know that the composition

$$\mathcal{L}_{n-1} \xrightarrow{R^{-1}} \mathcal{L}_n \xrightarrow{L} \mathcal{L}_{n-1}$$
$$v \longmapsto \underset{\substack{\uparrow \\ \text{uniquely} \\ \text{determined by } v}}{(v,*)} \longmapsto (v_2,\dots,v_{n-1},*)$$

is a bijection. By shift-invariance of X, it follows that every element in X is completely determined by the first $n-1$ symbols, therefore $|X|<\infty$. $\blacksquare$

End of lecture 9

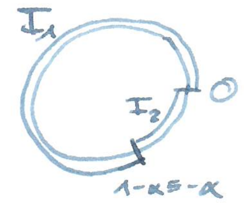
Definition.
| A Sturmian shift space is a shift X with $p_X(n)=n+1$ for all n (i.e., just not finite).

Remark: One can similarly define the complexity of a sequence by letting X be the shift space defined by a single sequence. One calls a sequence Sturmian if this associated shift space is Sturmian.
Similarly, one defines Sturmian one-sided shifts.
The Fibonacci numbers form a Sturmian sequence (Maybe exercise).

Constructions of Sturmian shifts

Example 0: $C_0 \longrightarrow \infty$

Example 1: Let $\alpha\in\mathbb{R}-\mathbb{Q}$. Define $I_1=[0,1-\alpha)\equiv[0,-\alpha)$ mod 1 and
$I_2=[1-\alpha,1)\equiv[-\alpha,0)$ mod 1.



We define the shift space by defining the language. To this end, we call a word w in $\{1,2\}^n$ allowed if

$$I_w := I_{w_1}\cap R_\alpha^{-1} I_{w_2}\cap\dots\cap R_\alpha^{-(n-1)} I_{w_n}\neq\emptyset,$$

i.e., there exists $x\in\mathbb{T}$ s.t.

$$\forall\, 0\le j<n \quad R_\alpha^j x\in I_{w_{j+1}}.$$

Claim:
| There are exactly $n+1$ allowed words of length n.

To this end it is useful to introduce the notation

$$P_k := \{R_\alpha^{-k} I_1, R_\alpha^{-k} I_2\},$$

i.e., for all $k\in\mathbb{N}$ the set $P_k\subseteq 2^{\mathbb{T}}$ is a partition of $\mathbb{T}$. We define for any two partitions $\mathcal{A},\mathcal{B}$ of $\mathbb{T}$ the common refinement

$$\mathcal{A}\vee\mathcal{B} := \{A\cap B : A\in\mathcal{A}, B\in\mathcal{B}\}.$$

Then $\mathcal{A}\vee\mathcal{B}$ is again a partition of $\mathbb{T}$.
Define inductively $Q_0 := P_0$ and for $k\in\mathbb{N}$

$$Q_k := Q_{k-1}\vee P_k.$$

Claim:
| $\{I_w : w\in\{1,2\}^n \text{ is allowed}\} = Q_{n-1}$

This is pretty much the definition.

Claim:
| $$\{w\in\{1,2\}^n \text{ is allowed}\} \longrightarrow Q_{n-1}$$
| $$w \longmapsto I_w$$
| is injective

Later