III – Measure preserving dynamics & Ergodic Theory

In what follows, we will always assume that $(X,\mathcal{B},\mu)$ is a probability space, i.e., X is a set, $\mathcal{B}$ is a σ-algebra on X, and $\mu:\mathcal{B}\to[0,1]$ is a probability measure.

In many examples, X is also a compact metric space and $\mathcal{B}$ is the Borel σ-algebra generated by the open sets.

Definition:

Let $(X,\mathcal{B},\mu)$ a probability space and let $(Y,\mathcal{C})$ a measurable space. If $T:X\to Y$ measurable, we denote by $T_*\mu$ the push-forward, i.e.,

$$\forall C\in\mathcal{C}\quad T_*\mu(C)=\mu(T^{-1}C).$$

A measurable map $T:X\to X$ is <u>measure preserving</u> if $T_*\mu=\mu$. In this case, we call μ a <u>T-invariant</u> measure and the system $(X,\mathcal{B},\mu,T)$ is a (probability) measure preserving system.

Lemma:

Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\sigma)$ probability spaces. The following are equivalent.

(i) $\sigma=T_*\mu$

(ii) $\forall f\geq 0$ measurable (from Y to $[0,\infty]$), we have

$$\int_Y f\,d\sigma=\int_X f\circ T\,d\mu.$$

(iii) $\forall f\in L^1_\sigma(Y)$

$$\int_Y f\,d\sigma=\int_X f\circ T\,d\mu.$$

Proof: Exercise (measure-theoretic induction).

Examples: (i) Let $(X,\mathcal{B},\mu)=(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathrm{Leb})$ and consider $T:X\to X$ given by the rotation R_α (for any $\alpha\in\mathbb{R}$). Then $(X,\mathcal{B},\mu,T)$ is a (probability) measure preserving system.

(ii) Let $p\in\mathbb{Z}$, then the multiplication by p defines a pmps $(\mathbb{T},\mathcal{B},\mathrm{Leb},T_p)$ (p non-zero).

(iii) Let G a compact metric group. Then G admits a (Borel) Haar measure m_G characterized uniquely by the following properties:

(H1) $\forall g\in G\ \forall B\in\mathcal{B}(G)$

$$m_G(gB)=m_G(B),$$

(H2) $\forall O\subset G$ open non-empty

$$m_G(O)>0,$$

(H3) $m_G(G)=1$. (this makes it unique)

Let $T:G\to G$ be a surjective endomorphism. Then $(G,\mathcal{B}(G),m_G,T)$ is a probability measure preserving system. (Exercise)

(iv) Let $A=\{0,\dots,p-1\}$ and $\underline{p}=(p_0,\dots,p_{p-1})$ a (fully supported) probability vector, i.e., $\forall 0\leq k<p$ $p_k\in(0,1]$ and $\sum_{k=0}^{p-1}p_k=1$.

Then $\underline{p}$ defines a probability measure $m_{\underline{p}}$ on $\mathcal{B}(A)=2^A$ by

$$\forall B\in\mathcal{B}(A)\quad m_{\underline{p}}(B)=\sum_{k\in B}p_k.$$

By the Kolmogorov extension theorem, there exists a unique probability measure $\mu_{\underline{p}}$ on $(A^{\mathbb{Z}},\mathcal{B}(A^{\mathbb{Z}}))$ s.t.

$\forall \ell\in\mathbb{N}\ \forall j_1,\dots,j_\ell\in\mathbb{Z}$ pairwise distinct

$$\mu_{\underline{p}}\Big(\underbrace{\{(x_n)_{n\in\mathbb{Z}}\in A^{\mathbb{Z}}:\forall 1\leq i\leq\ell\ x_{j_i}=a_i\}}_{{}_{j_1}[a_1]\cap\dots\cap{}_{j_\ell}[a_\ell]}\Big)=\prod_{i=1}^{\ell}m_{\underline{p}}(\{a_i\})=\prod_{i=1}^{\ell}p_{a_i}.$$

The latter determines $\mu_{\underline{p}}$ uniquely. Note that

$$\mu_{\underline{p}}(\sigma^{-1}({}_{j_1}[a_1]\cap\dots\cap{}_{j_\ell}[a_\ell]))=\mu_{\underline{p}}(\sigma^{-1}({}_{j_1}[a_1])\cap\dots\cap\sigma^{-1}({}_{j_\ell}[a_\ell]))$$

$$=\mu_{\underline{p}}({}_{j_1+1}[a_1]\cap\dots\cap{}_{j_\ell+1}[a_\ell])$$

$$=\prod_{i=1}^{\ell}p_{a_i}=\mu_{\underline{p}}({}_{j_1}[a_1]\cap\dots\cap{}_{j_\ell}[a_\ell])$$

Hence $(A^{\mathbb{Z}},\mathcal{B}(A^{\mathbb{Z}}),\mu_{\underline{p}},\sigma)$ is a probability measure preserving system.

One similarly obtains a pmps $(A^{\mathbb{N}},\mathcal{B}(A^{\mathbb{N}}),\mu^+_{\underline{p}},\sigma)$. These are called the two- and one-sided Bernoulli shifts on p symbols (for probability vector $\underline{p}$).

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(v) More generally, suppose that for all $\ell\in\mathbb{N}\cup\{0\}$ we are given a map

$$p^{(\ell)}:A^{\ell+1}\longrightarrow[0,1]$$

such that (i) $\sum_{a\in A}p^{(0)}(a)=1$

(ii) $\forall a_0,\dots,a_\ell\in A\quad p^{(\ell)}(a_0,\dots,a_\ell)=\sum_{a\in A}p^{(\ell+1)}(a_0,\dots,a_\ell,a)$.

Then there exists a unique probability measure μ on $\mathcal{B}(A^{\mathbb{Z}})$ such that
$\forall q\in\mathbb{Z}\,\forall \ell\geq 0\,\forall \underline{a}\in A^{\ell+1}$

$$\mu(\{x\in A^{\mathbb{Z}} : x_q=a_0,\dots,x_{q+\ell}=a_\ell\}) = p^{(\ell)}(\underline{a}).$$

This is again a consequence of Kolmogorov's extension theorem. One readily checks that this implies that $(A^{\mathbb{Z}},\mathcal{B}(A^{\mathbb{Z}}),\mu,\sigma)$ is a pmps.

<u>In class exercise:</u> How does this relate to example (iv)?

Here is an important example:

<u>(Finite state) Markov chains</u>

Let $\underline{p}\in\mathbb{R}^p$ be a probability vector and $\mathbb{P}=(p_{ij})_{i,j=0}^{p-1}$ a stochastic matrix, i.e.,

(i) $\forall i,j \quad p_{ij}\geq 0$

(ii) $\forall i \quad \sum_{j=0}^{p-1} p_{ij}=1$ (i.e., every row is a probability vector).

Also suppose that $\underline{p}$ is <u>$\mathbb{P}$-stationary</u> / $\mathbb{P}$ is a <u>transition matrix</u> for $\underline{p}$, i.e.,

$$\underline{p}\mathbb{P}=\underline{p}.$$

Define $p^{(\ell)}: A^{\ell+1}\to[0,1]$ by

$$p^{(\ell)}(a_0,\dots,a_\ell)=p_{a_0}p_{a_0a_1}\cdots p_{a_{\ell-1}a_\ell}.$$

Then the associated dynamical system

$$(A^{\mathbb{Z}},\mathcal{B}(A^{\mathbb{Z}}),\mu_{(\underline{p},\mathbb{P})},\sigma)$$

is the "two-sided $(\underline{p},\mathbb{P})$ Markov shift".

A special case is the Bernoulli shift $(p_{ij}=p_j)$.

(vi) <u>Gauss map / continued fraction map</u>

Let $Y=[0,1]-\mathbb{Q}$ and define $T: Y\to Y,\ y\mapsto \left\{\frac{1}{x}\right\}$.

<u>Theorem (Gauss)</u>

Define $\mu:\mathcal{B}([0,1])\to[0,1]$ by

$$\mu(B)=\frac{1}{\log 2}\int_B\frac{dx}{1+x}.$$

Then $T_*\mu=\mu$ and μ is a probability measure.

<u>Proof:</u> Exercise

Why is (vi) interesting?

Let $y\in Y$, then $T^n(y)\neq 0$ for all $n\geq 0$. Define

$$a_n(y):=\left\lfloor\frac{1}{T^{n-1}(y)}\right\rfloor \qquad (n\in\mathbb{N}).$$

Then
$$y=\lim_{n\to\infty}\cfrac{1}{a_1(y)+\cfrac{1}{a_2(y)+\cfrac{1}{a_3(y)+\dots+\cfrac{1}{a_n(y)}}}}$$

<u>Definition:</u>

Let $(X,\mathcal{B},\mu)$, $(Y,\mathcal{C},\sigma)$ be measure spaces and let $\varphi: X\to Y$ be a measurable map (defined μ-a.e.). φ is an invertible measure-preserving map if there exist $X'\in\mathcal{B}$ and $Y'\in\mathcal{C}$ s.t. $\mu(X')=\sigma(Y')=1$, $\varphi: X'\to Y'$ is measurable, bijective, and measure-preserving.

<u>Definition:</u>

Let $(X,\mathcal{B},\mu,T)$ and $(Y,\mathcal{C},\sigma,S)$ be pmps.

(1) $(Y,\mathcal{C},\sigma,S)$ is a (measurable) factor of $(X,\mathcal{B},\mu,T)$ if $\exists X'\in\mathcal{B}, Y'\in\mathcal{C}$ s.t. $1=\mu(X')=\sigma(Y')$, $TX'\subseteq X'$, $SY'\subseteq Y'$, and $\varphi: X'\to Y'$ measure preserving such that

$$\begin{tikzcd} X' \arrow[r, "T|_{X'}"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & X' \arrow[d, "\varphi"] \\ Y' \arrow[r, "S|_{Y'}"'] & Y' \end{tikzcd}$$

(2) $(Y,\mathcal{C},\sigma,S)$ and $(X,\mathcal{B},\mu,T)$ are isomorphic if $(Y,\mathcal{C},\sigma,S)$ is a factor s.t. the map φ (the factor map) can be chosen as an invertible measure-preserving map.

<u>Exercise</u> Check that $(X,\mathcal{B},\mu,T)\times(Y,\mathcal{C},\sigma,S)\longrightarrow(Y,\mathcal{C},\sigma,S)$ is a factor map.

<u>Example</u> Let $A=\{0,1\}$ and consider the one-sided Bernoulli shift induced by the uniform probability measure on A, i.e.,

$$\mathbb{X}=(A^{\mathbb{N}},\mathcal{B}(A^{\mathbb{N}}),(\tfrac{1}{2}\delta_{\{0\}}+\tfrac{1}{2}\delta_{\{1\}})^{\otimes\mathbb{N}},\sigma).$$

Let

$$\mathbb{Y}=(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathrm{Leb},T_2).$$

<u>Claim</u>

$\mathbb{X}\cong\mathbb{Y}$