Then there exists a unique probability measure μ on $\mathcal{B}(A^{\mathbb{Z}})$ such that $\forall q\in\mathbb{Z}\,\forall \ell\geq 0\,\forall \underline{a}\in A^{\ell+1}$

$$\mu(\{x\in A^{\mathbb{Z}} : x_q=a_0,\dots,x_{q+\ell}=a_\ell\}) = p^{(\ell)}(\underline{a}).$$

This is again a consequence of Kolmogorov's extension theorem. One readily checks that this implies that $(A^{\mathbb{Z}},\mathcal{B}(A^{\mathbb{Z}}),\mu,\sigma)$ is a pmps.

In class exercise: How does this relate to example (iv)?

Here is an important example:

(Finite state) Markov chains

Let $\underline{p}\in\mathbb{R}^P$ be a probability vector and $\mathbb{P}=(p_{ij})_{i,j=0}^{P-1}$ a stochastic matrix, i.e.,

(i) $\forall i,j$ $p_{ij}\geq 0$

(ii) $\forall i$ $\sum_{j=0}^{P-1} p_{ij}=1$ (i.e., every row is a probability vector).

Also suppose that $\underline{p}$ is $\mathbb{P}$-stationary / $\mathbb{P}$ is a transition matrix for $\underline{p}$, i.e.,

$$\underline{p}\,\mathbb{P}=\underline{p}.$$

Define $p^{(\ell)} : A^{\ell+1}\longrightarrow[0,1]$ by

$$p^{(\ell)}(a_0,\dots,a_\ell)=p_{a_0}p_{a_0a_1}\cdots p_{a_{\ell-1}a_\ell}.$$

Then the associated dynamical system

$$(A^{\mathbb{Z}},\mathcal{B}(A^{\mathbb{Z}}),\mu_{(\underline{p},\mathbb{P})},\sigma)$$

is the "two-sided $(\underline{p},\mathbb{P})$ Markov shift".

A special case is the Bernoulli shift ($p_{ij}=p_j$).

(vi) Gauss map / continued fraction map

Let $Y=[0,1]-\mathbb{Q}$ and define $T:Y\longrightarrow Y,$ $y\longmapsto\left\{\frac{1}{x}\right\}.$

Theorem (Gauss)

Define $\mu:\mathcal{B}([0,1])\longrightarrow[0,1]$ by

$$\mu(B)=\frac{1}{\log 2}\int_B\frac{dx}{1+x}.$$

Then $T_*\mu=\mu$ and μ is a probability measure.

Proof: Exercise

Why is (vi) interesting?

Let $y\in Y$, then $T^n(y)\neq 0$ for all $n\geq 0$. Define

$$a_n(y):=\left\lfloor\frac{1}{T^{n-1}(y)}\right\rfloor \qquad (n\in\mathbb{N}).$$

Then
$$y=\lim_{n\to\infty}\cfrac{1}{a_1(y)+\cfrac{1}{a_2(y)+\cfrac{1}{a_3(y)+\dots+\cfrac{1}{a_n(y)}}}}$$

End of lecture 11

III.1 - General notions

Definition:

Let $(X,\mathcal{B},\mu)$, $(Y,\mathcal{C},\sigma)$ be measure spaces and let $\varphi:X\longrightarrow Y$ be a measurable map (defined μ-a.e.). φ is an invertible measure-preserving map if there exist $X'\in\mathcal{B}$ and $Y'\in\mathcal{C}$ s.t. $\mu(X')=\sigma(Y')=1$, $\varphi:X'\longrightarrow Y'$ is measurable, bijective, and measure-preserving.

Definition:

Let $(X,\mathcal{B},\mu,T)$ and $(Y,\mathcal{C},\sigma,S)$ be pmps.

(1) $(Y,\mathcal{C},\sigma,S)$ is a (measurable) factor of $(X,\mathcal{B},\mu,T)$ if $\exists X'\in\mathcal{B}, Y'\in\mathcal{C}$ s.t. $1=\mu(X')=\sigma(Y')$, $TX'\subseteq X'$, $SY'\subseteq Y'$, and $\varphi:X'\longrightarrow Y'$ measure preserving such that

$$\begin{tikzcd} X' \arrow[r, "T|_{X'}"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & X' \arrow[d, "\varphi"] \\ Y' \arrow[r, "S|_{Y'}"'] & Y' \end{tikzcd}$$

(2) $(Y,\mathcal{C},\sigma,S)$ and $(X,\mathcal{B},\mu,T)$ are isomorphic if $(Y,\mathcal{C},\sigma,S)$ is a factor s.t. the map φ (the factor map) can be chosen as an invertible measure-preserving map.

Exercise Check that $(X,\mathcal{B},\mu,T)\times(Y,\mathcal{C},\sigma,S)\longrightarrow(Y,\mathcal{C},\sigma,S)$ is a factor map.

Example Let $A=\{0,1\}$ and consider the one-sided Bernoulli shift induced by the uniform probability measure on A, i.e.,

$$\mathbb{X}=(A^{\mathbb{N}},\mathcal{B}(A^{\mathbb{N}}),(\tfrac{1}{2}\delta_{\{0\}}+\tfrac{1}{2}\delta_{\{1\}})^{\otimes\mathbb{N}},\sigma).$$

Let

$$\mathbb{Y}=(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathrm{Leb},T_2).$$

Claim

$\mathbb{X}\cong\mathbb{Y}$

Step 1: $\mathbb{Y}$ is a factor of $\mathbb{X}$.

a) Let $\varphi: \{0,1\}^{\mathbb{N}} \longrightarrow \mathbb{T}$,

$$(x_n)_{n\in\mathbb{N}} \longmapsto \sum_{n\in\mathbb{N}} \frac{x_n}{2^n} \mod 1.$$

We have seen that this is well-defined and continuous and surjective.

b) Measure preservation:

Claim (clear)

$\mathcal{B}_{[0,1)}$ is generated by the collection

$$\left\{\left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right) : 0 \le k < 2^\ell, \ell \ge 0\right\} \quad (\text{"dyadic intervals"})$$

Note: A semi-algebra (over a set X) is a subset $\mathcal{S} \subseteq 2^X$ such that
(i) $\emptyset \in \mathcal{S}$
(ii) $\forall A, B \in \mathcal{S}$ $A \cap B \in \mathcal{S}$
(iii) $\forall A \in \mathcal{S}$ $X - A$ is a finite union of elements in $\mathcal{S}$

Therefore

$$\mathcal{S} = \left\{\left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right) : 0 \le k < 2^\ell, \ell \ge 0\right\} \cup \{\emptyset\}$$

is a semialgebra (over $[0,1)$).

Lemma:

Let $\lambda: \mathcal{S} \longrightarrow [0,\infty)$ be a countably additive measure, then there exists a unique extension of λ to a measure on $\sigma(\mathcal{S})$.

Proof cf. Appendix A in Einsiedler-Ward. λ extends to the algebra $\mathcal{A}$ generated by $\mathcal{S}$. Then one applies the monotone class theorem noting that the smallest monotone class containing $\mathcal{A}$ is $\sigma(\mathcal{S})$.

Algebra: semi-algebra s.t. $\forall A \in \mathcal{S}$ $X - A \in \mathcal{S}$

contains increasing unions and decreasing intersections

Hence it suffices to show that

$$\forall \ell \ge 0\ \forall 0 \le k < 2^\ell \quad \varphi_*\left(\frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}}\right)^{\mathbb{N}}\left(\left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right)\right) = \frac{1}{2^\ell}.$$

Note that

$$\varphi^{-1}\left(\left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right)\right) = \{(x_n) : x_1 = a_1, \dots, x_\ell = a_\ell\} \sqcup \{\underbrace{\bullet}_{\text{singleton}}\},$$

where $(a_1, \dots, a_\ell)$ is the reversed base-2 expansion of k, i.e.,

$$k = a_\ell + a_{\ell-1} \cdot 2 + \dots + a_1 \cdot 2^{\ell-1}.$$

We have seen this in the proof of forward transitivity of T_2.
The singleton comes from potentially replacing the last sequence

$(\dots, 1, 0, 0, \dots, 0)$

by

$(\dots, 0, 1, \dots, 1)$

This implies measure preservation.

Step 2: φ is an isomorphism

Lemma:

Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ distinct such that

$$\varphi((x_n)_n) = \varphi((y_n)_n).$$

Then $\exists N \in \mathbb{N}$

$$\forall k \in \mathbb{N} \quad x_N = x_{N+k} \text{ and } y_N = y_{N+k}.$$

More precisely, we have
(i) $\forall n < N$ $x_n = y_n$
(ii) $x_N \ne y_N$
(iii) If $x_N = 0$, then
$\forall n > N$ $x_n = 1 \wedge y_n = 0$.
If $x_N = 1$, then
$\forall n > N$ $x_n = 0 \wedge y_n = 1$.

Proof: Let $N = \inf\{n : x_n \ne y_n\}$. Then

$$0 = |\varphi(x) - \varphi(y)| = \left|\sum_{n \ge N} \frac{x_n - y_n}{2^n}\right|$$

$$\ge \left|\frac{1}{2^N} - \underbrace{\left|\sum_{n>N} \frac{x_n - y_n}{2^n}\right|}_{\le \frac{1}{2^N}}\right| = \frac{1}{2^N} - \left|\sum_{n>N} \frac{x_n - y_n}{2^n}\right| \ge 0$$

$$\Rightarrow \sum_{n>N} \frac{x_n - y_n}{2^n} = \pm \frac{1}{2^N} \Rightarrow |x_n - y_n| \text{ is constant equal to } 1.$$

Corollary:

φ is injective on

$$\{0,1\}^{\mathbb{N}} - \{(x_n)_{n\in\mathbb{N}} : x_n \text{ is eventually constant}\} =: X'$$

Note that $\{(x_n)_{n\in\mathbb{N}} : x_n \text{ is eventually constant}\}$ is a measurable nullset since the projections $(x_n)_n \longmapsto x_\ell$ are all continuous.
We are almost there: It remains to show that this set in the corollary surjects on the complement of a measurable nullset.

Lemma:

$$\varphi(X') = [0,1) - \mathbb{Z}\left[\tfrac{1}{2}\right] =: Y'$$

Proof: Suppose $t \notin \varphi(X')$. Let $(x_n) \in \{0,1\}^{\mathbb{N}}$ s.t. $t = \varphi(x_n)$. Thus

$$t = \sum_{n\in\mathbb{N}} \frac{x_n}{2^n} = \sum_{n=1}^{N} \frac{x_n}{2^n} \quad (\text{since w.l.o.g. } x_n = 0\ \forall n > N).$$

Step 3: Clearly $\sigma(X') = X'$ and $T_2(Y') = Y'$ (mod 1) and $\varphi \circ \sigma = T_2 \circ \varphi$ $\square$

III.2 – (Poincaré) recurrence

Theorem

Let $(X,\mathcal{B},\mu,T)$ a pmps. Let $E\in\mathcal{B}$. For μ-a.e. $x\in E$ there exists a sequence $(n_k)_{k\in\mathbb{N}}$ s.t. $n_k\nearrow\infty$ and $\forall k\in\mathbb{N}$ $T^{n_k}x\in E$.

Proof: Let $B=\{x\in E:\forall n\in\mathbb{N}\ T^n x\notin E\}$.

Claim

$$B=E\cap\left(\bigcap_{m\in\mathbb{N}}T^{-m}(X-E)\right)$$

This is clear.

$\therefore \forall n\in\mathbb{N}$ $\quad T^{-n}B=T^{-n}E\cap\left(\bigcap_{m\in\mathbb{N}}T^{-(m+n)}(X-E)\right).$

$\therefore \forall 0\le n_1<n_2 \qquad T^{-n_1}B\cap T^{-n_2}B=\emptyset.$

Since $(X,\mathcal{B},\mu,T)$ is a pmps, it follows that $\mu(B)=0$.

Let $F_1=E-B$, then $\mu(F_1)=\mu(E)$.

Now inductively construct F_2 using E,T^2 and so on. Then $F_k\subseteq E$ satisfies

$$\mu(F_k)=\mu(E),$$

hence $F=\bigcap_{k\in\mathbb{N}}F_k\subseteq E$ satisfies $\mu(F)=\mu(E)$ and every point in F returns to E infinitely often. $\boxtimes$

Corollary

Let (X,d) be a σ-compact metric space and let $\mathcal{B}(X)$ be the Borel σ-algebra on X. Suppose that $(X,\mathcal{B}(X),\mu,T)$ is a pmps. Then μ-a.e. $x\in X$ is recurrent, i.e., for μ-a.e. $x\in X$ there is a sequence $(n_k)_{k\in\mathbb{N}}$ s.t. $n_k\nearrow\infty$ and $\lim_{k\to\infty}T^{n_k}(x)=x$.

Remark: Probably/maybe σ-compactness is not needed.

Proof: Since X is σ-compact, for every $n\in\mathbb{N}$ there is a countable subset $X(n)=\{x_k^{(n)}:k\in\mathbb{N}\}$ such that $X=\bigcup_{k\in\mathbb{N}}B_{1/n}(x_k^{(n)})$. By Poincaré recurrence, for every $k,n\in\mathbb{N}$, there is a nullset $B(k,n)\subseteq B_{1/n}(x_k^{(n)})$ s.t. $\forall x\in B_{1/n}(x_k^{(n)})-B(k,n)$ the forward orbit of x returns to $B_{1/n}(x_k^{(n)})$ (infinitely often).

Let $N_n=\bigcup_{k\in\mathbb{N}}B(k,n)$, then N_n is a nullset, hence so is $N=\bigcup_{n\in\mathbb{N}}N_n$.

Claim:

Let $x\in X-N$, then x is recurrent.

Note that there is $k\in\mathbb{N}$ s.t. $x\in B_1(x_k^{(1)})$ and n_1 larger than 1 s.t. $T^{n_1}x\in B_1(x_k^{(1)})$.

Given $\ell\in\mathbb{N}$, let $k\in\mathbb{N}$ such that $x\in B_{1/(\ell+1)}(x_k^{(\ell+1)})$ and $n_{\ell+1}>n_\ell>\dots>n_1$ s.t. $T^{n_{\ell+1}}(x)\in B_{1/(\ell+1)}(x_k^{(\ell+1)})$ and so on.

Hence

$$d(x,T^{n_\ell}(x))\le d(x,x_k^{(\ell)})+d(x_k^{(\ell)},T^{n_\ell}(x))<\frac{2}{\ell}\xrightarrow{\ell\to\infty}0 \quad\boxtimes$$

III.3 – Associated isometries

Recall the following principle used already in algebra, topology, and calculus: Given a set X equipped with a structure $\mathcal{Z}$, e.g., $\mathcal{Z}$ might be a group structure, a topology, a manifold structure, i.e., a maximal atlas. In order to understand the pair $(X,\mathcal{Z})$, one has to understand functions on X which respect the structure $\mathcal{Z}$. For example, for vector spaces we were interested in linear functions. For groups, we are interested in (certain) group homomorphisms. For topological spaces, we are interested in continuous functions, and for manifolds we want to understand (germs of) smooth functions.

If $(X,\mathcal{B},\mu)$ is a probability space, there are many relevant function spaces.

Recall: • $L^0(X,\mathcal{B},\mu)=\{f:X\longrightarrow\mathbb{C}\cup\{\infty\}: f \text{ is measurable}\}/_\sim$, where

$$f_1\sim f_2 \Leftrightarrow \{x\in X: f_1(x)\neq f_2(x)\} \text{ is } \mu\text{-null}.$$

• Given $p\in[1,\infty)$

$$L^p(X,\mathcal{B},\mu)=\left\{f\in L^0(X,\mathcal{B},\mu): \int|f|^p\,d\mu<\infty\right\}/_\sim.$$

• $L^\infty(X,\mathcal{B},\mu)=\{f\in L^0(X,\mathcal{B},\mu): \exists\lambda>0\ |f(x)|<\lambda \text{ for } \mu\text{-a.e. } x\in X\}/_\sim.$

• $L^p_{\mathbb{R}}(X,\mathcal{B},\mu)=\{f\in L^p(X,\mathcal{B},\mu): f(x)\in\mathbb{R} \text{ for } \mu\text{-a.e. } x\in X\}/_\sim.$

• $L^p_+(X,\mathcal{B},\mu)=\{f\in L^p_{\mathbb{R}}(X,\mathcal{B},\mu): f(x)\ge 0 \text{ for } \mu\text{-a.e. } x\in X\}/_\sim.$

Definition:

Let $(X,\mathcal{B},\mu,T)$ a pmps. We denote

$$U_T: L^0(X,\mathcal{B},\mu)\longrightarrow L^0(X,\mathcal{B},\mu)$$
$$f\longmapsto f\circ T.$$

Remarks: • U_T is called the "Koopman operator" associated with $(X,\mathcal{B},\mu,T)$.

• Note that $U_T L^0_{\mathbb{R}}(X,\mathcal{B},\mu)\subseteq L^0_{\mathbb{R}}(X,\mathcal{B},\mu)$ and $U_T L^0_+(X,\mathcal{B},\mu)\subseteq L^0_+(X,\mathcal{B},\mu)$

Lemma

Let $(X, \mathcal{B}, \mu, T)$ a pmps. Then $\cdot\, U_T L^p(X,\mathcal{B},\mu) \subseteq L^p(X,\mathcal{B},\mu)$

$\cdot\, U_T L^p_{\mathbb{R}}(X,\mathcal{B},\mu) \subseteq L^p_{\mathbb{R}}(X,\mathcal{B},\mu)$

$\cdot\, U_T L^p_+(X,\mathcal{B},\mu) \subseteq L^p_+(X,\mathcal{B},\mu)$

Proof: One readily verifies that μ-a.e.

$$\mathrm{Re}(f\circ T) = \mathrm{Re}(f)\circ T, \quad \mathrm{Im}(f\circ T) = \mathrm{Im}(f)\circ T$$

and if $f \in L^p_{\mathbb{R}}(X,\mathcal{B},\mu)$

$$(f\circ T)_+ = f_+\circ T, \quad (f\circ T)_- = f_-\circ T.$$

Hence it suffices to show that $U_T L^p_+(X,\mathcal{B},\mu) \subseteq L^p_+(X,\mathcal{B},\mu)$.

Suppose first that $p=\infty$ and let $f \in L^\infty_+(X,\mathcal{B},\mu)$. Let $\lambda>0$ s.t. $f\in[0,\lambda]$ μ-a.e., then $\{x\in X : U_T f(x) > \lambda\} = T^{-1}\{x : f(x) > \lambda\}$

$$\therefore \mu(\{U_T f > \lambda\}) = \mu(T^{-1}\{f>0\}) = \mu(\{f>\lambda\}) = 0$$

$$\Rightarrow U_T f \in L^\infty_+(X,\mathcal{B},\mu).$$

Now suppose $p\in[1,\infty)$. If $f = \mathbb{1}_B$ for some $B\in\mathcal{B}$, then $U_T f = \mathbb{1}_{T^{-1}B}$ μ-a.e., hence $\int |U_T f|^p d\mu = \mu(T^{-1}B) = \mu(B) = \int |f|^p d\mu$.

If f is simple, i.e., f is a finite linear combination of indicator functions, then we can assume w.l.o.g. that the corresponding subsets are disjoint, i.e.,

$$f = \sum_{i=1}^r \alpha_i \mathbb{1}_{B_i},$$

where $1\le i<j\le r \Rightarrow B_i\cap B_j = \emptyset$. In particular

$$1\le i<j\le r \Rightarrow \mu(T^{-1}B_i \cap T^{-1}B_j) = \mu(T^{-1}(B_i\cap B_j)) = \mu(B_i\cap B_j) = 0.$$

Hence

$$\int |U_T f|^p d\mu = \int \Big|\sum_{i=1}^r \alpha_i \mathbb{1}_{T^{-1}B_i}\Big|^p d\mu = \sum_{i=1}^r \alpha_i^p \int \mathbb{1}^p_{T^{-1}B_i} d\mu$$

$$= \sum_{i=1}^r \alpha_i^p \int \mathbb{1}^p_{B_i} d\mu = \int \Big|\sum_{i=1}^r \alpha_i \mathbb{1}_{B_i}\Big|^p d\mu = \int |f|^p d\mu < \infty.$$

For general $f\in L^p_+(X,\mathcal{B},\mu)$, choose a sequence of simple functions $f_n \uparrow f$ and note that $U_T f_n \uparrow U_T f$ μ-a.e. Hence the claim follows from Levi's monotone convergence.

End of lecture 12

Corollary (of the proof)

Let $(X,\mathcal{B},\mu,T)$ a pmps. Then

$$U_T : L^2(X,\mathcal{B},\mu) \longrightarrow L^2(X,\mathcal{B},\mu)$$

is a unitary operator.

Remark: $\cdot$ Here we used the polarization identity:

$$\langle f,g\rangle = \int f\bar{g}\,d\mu = \frac{1}{4}\left(\|f+g\|_2^2 - \|f-g\|_2^2 + i\|f+ig\|_2^2 - i\|f-ig\|_2^2\right).$$

$\cdot$ Given $(X,\mathcal{B},\mu,T)$ a pmps, let

$$I_T = \{f\in L^2(X,\mathcal{B},\mu) : U_T f = f\}.$$

Then I_T is a closed subspace (and in part. $I_T = (I_T^\perp)^\perp$).

Theorem (von Neumann "Mean ergodic theorem")

Let $(X,\mathcal{B},\mu,T)$ a pmps and $P_T : L^2(X,\mathcal{B},\mu) \longrightarrow I_T$ the orthogonal projection. Given $f\in L^2(X,\mathcal{B},\mu)$, let $A_N f = \frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n$. Then

$$A_N f \xrightarrow{N\to\infty} P_T f \quad \text{in } L^2(X,\mathcal{B},\mu)$$

Proof: Let $B_T = \{f\in L^2(X,\mathcal{B},\mu) : \exists h\in L^2(X,\mathcal{B},\mu) \;\; f = U_T h - h\}$.

~~Claim:~~ $\cancel{I_T = B_T}$

Note: (i) Let $h\in L^2(X,\mathcal{B},\mu)$ and $f = U_T h - h$, then

$$\forall N\in\mathbb{N} \quad A_N f = \frac{1}{N}(h\circ T^N - h).$$

In particular, $\forall f\in B_T$ $A_N f \xrightarrow{N\to\infty} 0$ in $L^2(X,\mathcal{B},\mu)$.

(ii) Let $f\in I_T$, then

$$\forall N\in\mathbb{N} \quad A_N f = f = P_T f.$$

Claim:

$I_T = B_T^\perp$. In particular $L^2(X,\mathcal{B},\mu) = I_T \oplus \overline{B_T}$.

"$\subseteq$" Let $f\in I_T$ and $h\in L^2(X,\mathcal{B},\mu)$, then

$$\langle f, U_T h - h\rangle = \langle f, U_T h\rangle - \langle f,h\rangle = \langle U_T f, U_T h\rangle - \langle f,h\rangle = 0.$$

"$\supseteq$" Let $f\in B_T^\perp$, then $\forall h\in L^2(X,\mathcal{B},\mu)$

$$\langle h, U_T^* f\rangle = \langle U_T h, f\rangle = \langle h, f\rangle.$$