Lemma

Let $(X,\mathcal{B},\mu,T)$ a pmps. Then $\cdot\, U_T L^p(X,\mathcal{B},\mu) \subseteq L^p(X,\mathcal{B},\mu)$

$\cdot\, U_T L^p_{\mathbb{R}}(X,\mathcal{B},\mu) \subseteq L^p_{\mathbb{R}}(X,\mathcal{B},\mu)$

$\cdot\, U_T L^p_+(X,\mathcal{B},\mu) \subseteq L^p_+(X,\mathcal{B},\mu)$

Proof: One readily verifies that μ-a.e.

$$\mathrm{Re}(f\circ T) = \mathrm{Re}(f)\circ T, \quad \mathrm{Im}(f\circ T) = \mathrm{Im}(f)\circ T$$

and if $f \in L^p_{\mathbb{R}}(X,\mathcal{B},\mu)$

$$(f\circ T)_+ = f_+\circ T, \quad (f\circ T)_- = f_-\circ T.$$

Hence it suffices to show that $U_T L^p_+(X,\mathcal{B},\mu) \subseteq L^p_+(X,\mathcal{B},\mu)$.

Suppose first that $p=\infty$ and let $f\in L^\infty_+(X,\mathcal{B},\mu)$. Let $\lambda>0$ s.t. $f\in[0,\lambda]$ μ-a.e., then $\{x\in X: U_T f(x)>\lambda\} = T^{-1}\{x: f(x)>\lambda\}$

$$\therefore \mu(\{U_T f>\lambda\}) = \mu(T^{-1}\{f>0\}) = \mu(\{f>\lambda\}) = 0$$

$$\Rightarrow U_T f \in L^\infty_+(X,\mathcal{B},\mu).$$

Now suppose $p\in[1,\infty)$. If $f = \mathbb{1}_B$ for some $B\in\mathcal{B}$, then $U_T f = \mathbb{1}_{T^{-1}B}$ μ-a.e., hence $\int |U_T f|^p d\mu = \mu(T^{-1}B) = \mu(B) = \int |f|^p d\mu$.

If f is simple, i.e., f is a finite linear combination of indicator functions, then we can assume w.l.o.g. that the corresponding subsets are disjoint, i.e.,

$$f = \sum_{i=1}^r \alpha_i \mathbb{1}_{B_i},$$

where $1\le i<j\le r \Rightarrow B_i\cap B_j = \emptyset$. In particular

$$1\le i<j\le r \Rightarrow \mu(T^{-1}B_i \cap T^{-1}B_j) = \mu(T^{-1}(B_i\cap B_j)) = \mu(B_i\cap B_j) = 0.$$

Hence

$$\int |U_T f|^p d\mu = \int \Big|\sum_{i=1}^r \alpha_i \mathbb{1}_{T^{-1}B_i}\Big|^p d\mu = \sum_{i=1}^r \alpha_i^p \int \mathbb{1}^p_{T^{-1}B_i} d\mu$$
$$= \sum_{i=1}^r \alpha_i^p \int \mathbb{1}^p_{B_i} d\mu = \int \Big|\sum_{i=1}^r \alpha_i \mathbb{1}_{B_i}\Big|^p d\mu = \int |f|^p d\mu < \infty.$$

For general $f\in L^p_+(X,\mathcal{B},\mu)$, choose a sequence of simple functions $f_n \nearrow f$ and note that $U_T f_n \nearrow U_T f$ μ-a.e. Hence the claim follows from Levi's monotone convergence.

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Corollary (of the proof)

Let $(X,\mathcal{B},\mu,T)$ a pmps. Then

$$U_T : L^2(X,\mathcal{B},\mu) \longrightarrow L^2(X,\mathcal{B},\mu)$$

is an isometry.

Remark: Here we used the polarization identity:

or directly: $\langle U_T f, U_T g\rangle_2 = \int_X (f\cdot\bar g)\circ T\, d\mu = \int_X f\cdot \bar g\, dT_*\mu = \int_X f\cdot\bar g\, d\mu = \langle f,g\rangle_2$

$$\langle f,g\rangle = \int f\bar g\, d\mu = \frac{1}{4}\left(\|f+g\|_2^2 - \|f-g\|_2^2 + i\|f+ig\|_2^2 - i\|f-ig\|_2^2\right).$$

- Given $(X,\mathcal{B},\mu,T)$ a pmps, let

$$I_T = \{f\in L^2(X,\mathcal{B},\mu) : U_T f = f\}.$$

Then I_T is a closed subspace (and in part. $I_T = (I_T^\perp)^\perp$).

Theorem (von Neumann "Mean ergodic theorem")

Let $(X,\mathcal{B},\mu,T)$ a pmps and $P_T : L^2(X,\mathcal{B},\mu) \longrightarrow I_T$ the orthogonal projection. Given $f\in L^2(X,\mathcal{B},\mu)$, let $A_N f = \frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n$. Then

$$A_N f \xrightarrow{N\to\infty} P_T f \quad \text{in } L^2(X,\mathcal{B},\mu)$$

Proof: Let $B_T = \{f\in L^2(X,\mathcal{B},\mu) : \exists h\in L^2(X,\mathcal{B},\mu)\ \ f = U_T h - h\}$.

~~Claim:~~

Note: (i) Let $h\in L^2(X,\mathcal{B},\mu)$ and $f = U_T h - h$, then

$$\forall N\in\mathbb{N}\quad A_N f = \frac{1}{N}(h\circ T^N - h).$$

In particular, $\forall f\in B_T$ $A_N f \xrightarrow{N\to\infty} 0$ in $L^2(X,\mathcal{B},\mu)$.

(ii) Let $f\in I_T$, then

$$\forall N\in\mathbb{N}\ A_N f = f = P_T f.$$

Claim:

$I_T = B_T^\perp$. In particular $L^2(X,\mathcal{B},\mu) = I_T \oplus \overline{B_T}$.

"$\subseteq$" Let $f\in I_T$ and $h\in L^2(X,\mathcal{B},\mu)$, then

$$\langle f, U_T h - h\rangle = \langle f, U_T h\rangle - \langle f,h\rangle = \langle U_T f, U_T h\rangle - \langle f,h\rangle = 0.$$

"$\supseteq$" Let $f\in B_T^\perp$, then $\forall h\in L^2(X,\mathcal{B},\mu)$

$$\langle h, U_T^* f\rangle = \langle U_T h, f\rangle = \langle h, f\rangle.$$

Hence $U_T^* f = f$ (since we can choose $h = U_T^* f - f$).

Thus

$$\|U_T f - f\|_2^2 = 2\|f\|_2^2 - 2\mathrm{Re}(\langle U_T f, f\rangle) = 2\|f\|_2^2 - 2\mathrm{Re}(\underbrace{\langle f, U_T^* f\rangle}_{=\|f\|_2^2}) = 0.$$

Claim

Let $g \in \overline{B_T}$, then $A_N g \xrightarrow{N\to\infty} 0$.

Choose $\varepsilon > 0$ arbitrary and let $\tilde g \in B_T$ s.t. $\|g - \tilde g\|_2 < \varepsilon$. Then

$$\|A_N g\|_2 \le \underbrace{\|A_N g - A_N \tilde g\|_2}_{<\varepsilon} + \underbrace{\|A_N \tilde g\|_2}_{\to 0}$$

$\Rightarrow$ For sufficiently large $N \in \mathbb{N}$ we have

$$\|A_N g\|_2 \le \varepsilon.$$

$\boxtimes$

III.4 - Ergodicity

Definition

Let $(X, \mathcal{B}, \mu, T)$ a pmps. We call $(X, \mathcal{B}, \mu, T)$ (or μ or T) *ergodic* if

$$\{f \in L^2(X, \mathcal{B}, \mu, T) : U_T f = f\} = \{\alpha \mathbb{1}_X : \alpha \in \mathbb{C}\}.$$

Proposition

Let $(X, \mathcal{B}, \mu, T)$ be a pmps. The following are equivalent.

(i) T is ergodic.

(ii) $\forall B \in \mathcal{B}$

$$T^{-1}B = B \Rightarrow \mu(B) \in \{0,1\}.$$

(iii) $\forall B \in \mathcal{B}$

$$\mu(T^{-1}B \Delta B) = 0 \Rightarrow \mu(B) \in \{0,1\}.$$

$A \Delta B = A \cup B - A \cap B$

(iv) $\forall B \in \mathcal{B}$

$$\mu(B) > 0 \Rightarrow \mu\Big(\bigcup_{n\in\mathbb{N}} T^{-n} B\Big) = 1.$$

(v) $\forall A, B \in \mathcal{B}$

$$\mu(A)\mu(B) > 0 \Rightarrow \exists n \in \mathbb{N}\ \mu(T^{-n}A \cap B) > 0.$$

(vi) $\forall f \in L^0(X, \mathcal{B}, \mu)$

$$U_T f = f \Rightarrow \exists c \in \mathbb{C}\ f = c\mathbb{1}_X \ (\text{in } L^0(X, \mathcal{B}, \mu)).$$

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Proof:

$$\begin{tikzcd}
(vi) \arrow[rr, Rightarrow] & & (i) \arrow[dl, Rightarrow] \\
& (iii) \arrow[ul, Rightarrow] \arrow[dr, Rightarrow] & \\
(ii) \arrow[ur, Rightarrow] & & (iv) \arrow[dl, Rightarrow] \\
& (v) \arrow[ul, Rightarrow] &
\end{tikzcd}$$

(vi) $\Rightarrow$ (i): This is immediate.

(i) $\Rightarrow$ (iii): Suppose $\mu(T^{-1}B \Delta B) = 0$, i.e., $\mathbb{1}_{T^{-1}B\Delta B} = 0$ μ-a.e., i.e.,

$$\mathbb{1}_{T^{-1}B \cup B} = \mathbb{1}_{T^{-1}B \cap B} \quad \mu\text{-a.e.}$$

$$\| $$

$$\mathbb{1}_{T^{-1}B} + \mathbb{1}_B - \mathbb{1}_{T^{-1}B\cap B}$$

$$\Rightarrow \mathbb{1}_{T^{-1}B} + \mathbb{1}_B \in \{0,2\} \quad \mu\text{-a.e.}$$

$$\Rightarrow U_T \mathbb{1}_B = \mathbb{1}_B \quad \mu\text{-a.e.}$$

$$\Rightarrow \mathbb{1}_B = c \cdot \mathbb{1}_X \text{ for some } c \in \mathbb{C} \quad \mu\text{-a.e.}$$

(iii) $\Rightarrow$ (iv): Let $E = \bigcup_{n\in\mathbb{N}} T^{-n}B$. Then $T^{-1}E \subseteq E$. Since $\mu(T^{-1}E) = \mu(E)$, we have $T^{-1}E \Delta E = E - T^{-1}E$ and, thus, $\mu(T^{-1}E\Delta E) = 0$

$$\Rightarrow \mu(E) \in \{0,1\}.$$

But $\mu(E) \ge \mu(T^{-1}B) = \mu(B) > 0$ implies the claim.

(iv) $\Rightarrow$ (v): Suppose $\mu(A)\mu(B) > 0$, so that $\mu\Big(\bigcup_{n\in\mathbb{N}} T^{-n}A\Big) = 1$

$$\Rightarrow \mu(B) = \mu\Big(B \cap \bigcup_{n\in\mathbb{N}} T^{-n}A\Big) \le \sum_{n\in\mathbb{N}} \mu(T^{-n}A \cap B)$$

$$\Rightarrow \exists n \in \mathbb{N}\ \mu(T^{-n}A \cap B) > 0.$$

(v) $\Rightarrow$ (ii): Suppose $T^{-1}B = B$. Then

$$0 = \mu(B \cap B^c) = \mu(T^{-1}B \cap B^c) \overset{\forall n\in\mathbb{N}}{=} \mu(T^{-n}B \cap B^c)$$

$$\Rightarrow \mu(B)\mu(B^c) = 0.$$

(ii) $\Rightarrow$ (iii): Suppose $\mu(T^{-1}B\Delta B) = 0$. We need to construct $C \in \mathcal{B}$ such that $\mu(C) = \mu(B)$ and $T^{-1}C = C$. We'll show that we can use

$$C = \limsup_n T^{-n}B := \bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} T^{-n}B.$$