Hence $U_T^* f = f$ (since we can choose $h = U_T^* f - f$).

Thus
$$\|U_T f - f\|_2^2 = 2\|f\|_2^2 - 2\mathrm{Re}(\langle U_T f, f\rangle) = 2\|f\|_2^2 - 2\mathrm{Re}(\underbrace{\langle f, U_T^* f\rangle}_{=\|f\|_2^2}) = 0.$$

Claim

Let $g \in \overline{B_T}$, then $A_N g \xrightarrow{N\to\infty} 0$.

Choose $\varepsilon > 0$ arbitrary and let $\tilde{g} \in B_T$ s.t. $\|g - \tilde{g}\|_2 < \varepsilon$. Then
$$\|A_N g\|_2 \leq \underbrace{\|A_N g - A_N \tilde{g}\|_2}_{<\varepsilon} + \underbrace{\|A_N \tilde{g}\|_2}_{\to 0}$$

$\Rightarrow$ For sufficiently large $N \in \mathbb{N}$ we have
$$\|A_N g\|_2 \leq \varepsilon.$$

$\blacksquare$

III.4 - Ergodicity

Definition

Let $(X, \mathcal{B}, \mu, T)$ a pmps. We call $(X, \mathcal{B}, \mu, T)$ (or μ or T) <u>ergodic</u> if
$$\{f \in L^2(X, \mathcal{B}, \mu, T) : U_T f = f\} = \{\alpha \mathbb{1}_X : \alpha \in \mathbb{C}\}.$$

Proposition

Let $(X, \mathcal{B}, \mu, T)$ be a pmps. The following are equivalent.

(i) T is ergodic.

(ii) $\forall B \in \mathcal{B}$
$$T^{-1}B = B \Rightarrow \mu(B) \in \{0,1\}.$$

(iii) $\forall B \in \mathcal{B}$
$$\mu(T^{-1}B \Delta B) = 0 \Rightarrow \mu(B) \in \{0,1\}.$$

$A \Delta B = A \cup B - A \cap B$

(iv) $\forall B \in \mathcal{B}$
$$\mu(B) > 0 \Rightarrow \mu\Big(\bigcup_{n\in\mathbb{N}} T^{-n} B\Big) = 1.$$

(v) $\forall A, B \in \mathcal{B}$
$$\mu(A)\mu(B) > 0 \Rightarrow \exists n \in \mathbb{N}\ \mu(T^{-n}A \cap B) > 0.$$

(vi) $\forall f \in L^0(X, \mathcal{B}, \mu)$
$$U_T f = f \Rightarrow \exists c \in \mathbb{C}\ f = c\mathbb{1}_X \ (\text{in } L^0(X, \mathcal{B}, \mu)).$$

End of lecture 13

Proof:
$$\begin{tikzcd}
(vi) \arrow[rr, Rightarrow] & & (i) \arrow[dl, Rightarrow] \\
& (iii) \arrow[ul, Rightarrow] \arrow[dr, Rightarrow] & \\
(ii) \arrow[ur, Rightarrow] & & (iv) \arrow[dl, Rightarrow] \\
& (v) \arrow[ul, Rightarrow] &
\end{tikzcd}$$

<u>(vi) $\Rightarrow$ (i):</u> This is immediate.

<u>(i) $\Rightarrow$ (iii):</u> Suppose $\mu(T^{-1}B \Delta B) = 0$, i.e., $\mathbb{1}_{T^{-1}B\Delta B} = 0$ μ-a.e., i.e.,
$$\mathbb{1}_{T^{-1}B \cup B} = \mathbb{1}_{T^{-1}B \cap B} \quad \mu\text{-a.e.}$$
$$\|$$
$$\mathbb{1}_{T^{-1}B} + \mathbb{1}_B - \mathbb{1}_{T^{-1}B\cap B}$$

$$\Rightarrow \mathbb{1}_{T^{-1}B} + \mathbb{1}_B \in \{0,2\} \quad \mu\text{-a.e.}$$
$$\Rightarrow U_T \mathbb{1}_B = \mathbb{1}_B \quad \mu\text{-a.e.}$$
$$\Rightarrow \mathbb{1}_B = c\cdot\mathbb{1}_X \text{ for some } c \in \mathbb{C} \quad \mu\text{-a.e.}$$

<u>(iii) $\Rightarrow$ (iv):</u> Let $E = \bigcup_{n\in\mathbb{N}} T^{-n}B$. Then $T^{-1}E \subseteq E$. Since $\mu(T^{-1}E) = \mu(E)$, we have $T^{-1}E \Delta E = E - T^{-1}E$ and, thus, $\mu(T^{-1}E \Delta E) = 0$
$$\Rightarrow \mu(E) \in \{0,1\}.$$
But $\mu(E) \geq \mu(T^{-1}B) = \mu(B) > 0$ implies the claim.

<u>(iv) $\Rightarrow$ (v):</u> Suppose $\mu(A)\mu(B) > 0$, so that $\mu\Big(\bigcup_{n\in\mathbb{N}} T^{-n}A\Big) = 1$
$$\Rightarrow \mu(B) = \mu\Big(B \cap \bigcup_{n\in\mathbb{N}} T^{-n}A\Big) \leq \sum_{n\in\mathbb{N}} \mu(T^{-n}A \cap B)$$
$$\Rightarrow \exists n \in \mathbb{N}\ \mu(T^{-n}A \cap B) > 0.$$

<u>(v) $\Rightarrow$ (ii):</u> Suppose $T^{-1}B = B$. Then
$$0 = \mu(B \cap B^c) = \mu(T^{-1}B \cap B^c) \overset{\forall n\in\mathbb{N}}{=} \mu(T^{-n}B \cap B^c)$$
$$\Rightarrow \mu(B)\mu(B^c) = 0.$$

<u>(ii) $\Rightarrow$ (iii):</u> Suppose $\mu(T^{-1}B \Delta B) = 0$. We need to construct $C \in \mathcal{B}$ such that $\mu(C) = \mu(B)$ and $T^{-1}C = C$. We'll show that we can use
$$C = \limsup_n T^{-n}B := \bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} T^{-n}B.$$

For what follows, let $C_N = \bigcup_{n=N}^{\infty} T^{-n}B$ and note that $C = \bigcap_{N=0}^{\infty} C_N$, $C_0 \supseteq C_1 \supseteq \cdots$.

Fact:

Let $A, B, C \subseteq X$, then $A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$

Indeed, suppose $x \in A \Delta C \Rightarrow x \in A \cup C$ and $x \notin A \cap C$.

Case 1: $x \in B$

In this case $x \notin A \cap C \Rightarrow x \in B - A$ or $x \in B - C \Rightarrow x \in (A \Delta B) \cup (B \Delta C)$

Case 2: $x \notin B$

In this case $x \in A \cup C \Rightarrow x \in A - B$ or $x \in C - B \Rightarrow x \in (A \Delta B) \cup (B \Delta C)$

The fact implies by induction that

$$\forall n \in \mathbb{N} \quad B \Delta T^{-n} B \subseteq \bigcup_{i=0}^{n-1} (T^{-i} B \Delta T^{-(i+1)} B).$$

Claim: For all $N \geq 0$

$\mu(B \Delta C_N) = 0$

Pf of claim

We first note that $B \Delta C_N \subseteq \bigcup_{n=N}^{\infty} (B \Delta T^{-n} B)$. To this end suppose that $x \in B \Delta C_N$. If $x \in B$, then $\forall n \geq N$ $x \notin T^{-n}B$ and, thus, for all $n \in \mathbb{N}$ $x \in B \Delta T^{-n} B$. If $x \notin B$, then $x \in C_N$ and thus there is $n \geq N$ s.t. $x \in T^{-n}B$. Hence $x \in B \Delta T^{-n} B$.

It follows that $B \Delta C_N \subseteq \bigcup_{n=N}^{\infty} (B \Delta T^{-n} B)$.

Now this yields

$$\mu(B \Delta C_N) \leq \sum_{n=N}^{\infty} \mu(B \Delta T^{-n} B) \leq \sum_{n=N}^{\infty} \sum_{i=0}^{n-1} \mu(T^{-i} B \Delta T^{-(i+1)} B)$$

$$= \sum_{n=N}^{\infty} \sum_{i=0}^{n-1} \underbrace{\mu(T^{-i}(B \Delta T^{-1} B))}_{=0} = 0.$$

Claim:

$\mu(B \Delta C) = 0$

Pf of claim

Note that $B \Delta C \subseteq \bigcup_{N \geq 0} (C_N \Delta B)$. We argue as above: Let $x \in B \cup C$.

Case 1: $x \in B$, then $x \notin C \Rightarrow \exists N \in \mathbb{N} \cup \{0\}$ s.t. $x \notin C_N \Rightarrow x \in B - C_N \subseteq B \Delta C_N$.

Case 2: $x \notin B$, then $x \in C_N$ for all $N \in \mathbb{N} \cup \{0\} \Rightarrow \exists N \in \mathbb{N} \cup \{0\}$ s.t. $x \in C_N - B \subseteq B \Delta C_N$.

Hence

$$\mu(B \Delta C) \leq \sum_{N \geq 0} \mu(B \Delta C_N) = 0.$$

Suppose $\mu(C) > \mu(B)$, then $\mu(B \cup C) \geq \mu(C) > \mu(B) \geq \mu(B \cap C)$ $\Rightarrow \mu(B \Delta C) = \mu(B \cup C) - \mu(B \cap C) > 0$ ↯

It follows that $\mu(B) = \mu(C)$.

Moreover

$$T^{-1}C = \bigcap_{N=0}^{\infty} T^{-1} C_N = \bigcap_{N=0}^{\infty} C_{N+1} = C.$$

Hence

$$\mu(B) = \mu(C) \in \{0, 1\}.$$

Note: We have shown that $\forall B \in \mathcal{B}$ s.t. $\mu(B \Delta T^{-1}B) = 0$ there is $C \in \mathcal{B}$ such that $T^{-1}C = C$ and $\mu(C \Delta B) = 0$

(iii) $\Rightarrow$ (vi): Suppose $f \in L^0(X, \mathcal{B}, \mu)$ and $U_T f = f$. Then

$$U_T \mathrm{Re}(f) = \mathrm{Re}(U_T f) = \mathrm{Re}(f) \quad \mu\text{-a.e.}$$

$$U_T \mathrm{Im}(f) = \mathrm{Im}(U_T f) = \mathrm{Im}(f) \quad \mu\text{-a.e.}$$

Hence $f \in L^0_{\mathbb{R}}(X, \mathcal{B}, \mu)$ without loss of generality.

Given $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$A_n^k = f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right).$$

Then $X = \bigcup_{k \in \mathbb{Z}} A_n^k$ and $T^{-1} A_n^k \Delta A_n^k \subseteq \{U_T f \neq f\}$, hence

$$\forall n \in \mathbb{N} \forall k \in \mathbb{Z} \quad \mu(A_n^k) \in \{0, 1\}.$$

$$\therefore \forall n \in \mathbb{N} \exists! k_n \in \mathbb{Z} \quad \mu(A_n^{k_n}) = 1.$$

Then

$$\mu\left(\bigcap_{n \in \mathbb{N}} A_n^{k_n}\right) = 1$$

and f is constant on $\bigcap_{n \in \mathbb{N}} A_n^{k}$.

∎

III.5 – The pointwise ergodic theorem

We next want to prove a considerable strengthening of von Neumann's mean ergodic theorem, namely the pointwise ergodic theorem due to Birkhoff. We have seen before that ergodicity means that the invariant (or "almost invariant") sets are "measurably trivial", i.e., have either measure zero or measure one. One easily verifies that the invariant and the almost invariant sets form a σ-algebra.

Proposition

Let $(X,\mathcal{B},\mu,T)$ a pmps. Then

$$\mathcal{E}_T = \{B\in\mathcal{B} : T^{-1}B = B\}$$

$$\mathcal{E}_{T,\mu} = \{B\in\mathcal{B} : \mu(T^{-1}B\,\Delta\, B)=0\}$$

are σ-algebras. Moreover, for any $B\in\mathcal{E}_{T,\mu}$ there is $C\in\mathcal{E}_T$ such that

$$\mu(B\Delta C)=0.$$

Proof: The first two claims are immediate. The last claim was proven when we discussed the equivalent characterizations of ergodicity, i.e., suppose that $B\in\mathcal{E}_{T,\mu}$, then

$$C=\limsup_{n\to\infty} T^{-n}B\in\mathcal{E}_T$$

and $\mu(B\Delta C)=0$. ∎

An excursion into probability theory:

Proposition (Conditional Expectation)

Let $(X,\mathcal{B},\mu,T)$ be a probability space and let $\mathcal{A}\subseteq\mathcal{B}$ be a σ-algebra. There exists a linear operator (bounded)

$$\mathbb{E}(\cdot\,|\mathcal{A}) : L^1(X,\mathcal{B},\mu)\longrightarrow L^1(X,\mathcal{B},\mu)$$

such that the following are true.

(*) $\forall f\in L^1(X,\mathcal{B},\mu)$ (a representative of) $\mathbb{E}(f|\mathcal{A})$ is $\mathcal{A}$-measurable.

(**) $\forall f\in L^1(X,\mathcal{B},\mu)\,\forall A\in\mathcal{A}$

$$\int_A \mathbb{E}(f|\mathcal{A})\,d\mu = \int_A f\,d\mu.$$

Moreover, for any $\mathcal{A}$-measurable $\varphi\in L^1(X,\mathcal{B},\mu)$ and for any $f\in L^1(X,\mathcal{B},\mu)$ we have

$$\forall A\in\mathcal{A}\ \int_A\varphi\,d\mu=\int_A f\,d\mu \implies \mathbb{E}(f|\mathcal{A})=\varphi.$$

We will give a proof using that

- $L^2(X,\mathcal{A},\mu)\subseteq L^2(X,\mathcal{B},\mu)$ is a closed subspace,
- $L^2(X,\mathcal{B},\mu)\subseteq L^1(X,\mathcal{B},\mu)$ is dense, and
- if $P_{\mathcal{A}} : L^2(X,\mathcal{B},\mu)\longrightarrow L^2(X,\mathcal{A},\mu)$ is the orthogonal projection, then $P_{\mathcal{A}}$ is continuous with respect to the L^1-norm and, hence, admits a unique extension $\mathbb{E}(\cdot|\mathcal{A}) : L^1(X,\mathcal{B},\mu)\longrightarrow L^1(X,\mathcal{B},\mu)$.

A detour on orthogonal projections for Hilbert spaces:

Lemma:

Let $\mathcal{H}$ be a Hilbert space, i.e., $\mathcal{H}$ admits an inner product and $\mathcal{H}$ is complete with respect to the induced metric. Let $C\subseteq\mathcal{H}$ be a non-empty convex closed subset. Then $\exists x\in C$ such that

$$\|x\| = \inf\{\|v\| : v\in C\}.$$

Proof: Let $\delta=\inf\{\|v\|:v\in C\}$ and choose $(v_n)_{n\in\mathbb{N}}\in C^{\mathbb{N}}$ such that $\delta=\lim_{n\to\infty}\|v_n\|$. Since C is convex, for any $m,n\in\mathbb{N}$ we have that $\left\|\frac{v_n+v_m}{2}\right\|\geq\delta$, hence by the parallelogram law for inner product spaces, for all $n,m\in\mathbb{N}$ we have

$$\|v_n-v_m\|^2 = 2\|v_n\|^2+2\|v_m\|^2-4\left\|\frac{v_n+v_m}{2}\right\|^2$$

$$\leq 2\|v_n\|^2+2\|v_m\|^2-4\delta^2.$$

Since $\|v_n\|^2,\|v_m\|^2 \xrightarrow{n,m\to\infty}\delta^2$, it follows that $(v_n)_{n\in\mathbb{N}}$ is Cauchy and, since $\mathcal{H}$ is complete and C is closed, $v=\lim_{n\to\infty} v_n\in C$. ∎

Lemma:

Let $V\subseteq\mathcal{H}$ a closed subspace, $x\in\mathcal{H}$, then $V-x=\{v-x : v\in V\}$ is a non-empty closed convex subset of $\mathcal{H}$.

Proof: $-x\in V-x$, hence non-empty. Translation by $-x$ is an isometry, hence continuous with continuous inverse, thus $V-x$ is closed. Finally, suppose $v,w\in V$ and $t\in[0,1]$, then $t(v-x)+(1-t)(w-x) = (tv+(1-t)w)-x\in V-x$ since V is convex.

Corollary:

Let $V\subseteq\mathcal{H}$ closed subspace, $x\in\mathcal{H}$, then there is $Px\in V$ s.t. $\|x-Px\| = \inf\{\|x-v\| : v\in V\}$.

<u>Lemma</u>

Let $V \subseteq \mathcal{H}$ a closed subspace, $x \in \mathcal{H}$, and Px as in the preceding corollary. Then $x - Px \in V^\perp$ and this characterizes Px uniquely among elements in V.

<u>Proof</u>: Let $v \in V$. We want to show that $\langle x - Px, v\rangle = 0$. If $v = 0$, this is clear. So suppose $v \neq 0$ and let $\lambda \in \mathbb{C}$ arbitrary (to be chosen cleverly later). Then

$$\|x - Px\|^2 \le \|x - Px - \lambda v\|^2 = \|x - Px\|^2 - 2\mathrm{Re}(\langle x - Px, \lambda v\rangle) + |\lambda|^2\|v\|^2,$$

therefore

$$2\mathrm{Re}(\langle x - Px, \lambda v\rangle) \le |\lambda|^2 \|v\|^2.$$

Choose $\lambda = \frac{\langle x - Px, v\rangle}{\|v\|^2}$, then $\mathrm{Re}(\langle x - Px, \lambda v\rangle) = \mathrm{Re}\left(\frac{|\langle x - Px, v\rangle|^2}{\|v\|^2}\right) = \frac{|\langle x - Px, v\rangle|^2}{\|v\|^2}$,

hence

$$2\frac{|\langle x - Px, v\rangle|^2}{\|v\|^2} = 2\mathrm{Re}(\langle x - Px, \lambda v\rangle) \le |\lambda|^2\|v\|^2 = \frac{|\langle x - Px, v\rangle|^2}{\|v\|^4}\|v\|^2 = \frac{|\langle x - Px, v\rangle|^2}{\|v\|^2},$$

thus $\langle x - Px, v\rangle = 0$.

For the uniqueness, let $v^* \in V$ such that $x - v^* \in V^\perp$, then

$$V \ni Px - v^* = (x - v^*) - (x - Px) \in V^\perp,$$

thus $Px - v^* = 0$. $\ulcorner y \in V \cap V^\perp \Rightarrow \langle y, y\rangle = 0 \lrcorner$

$\boxtimes$

<u>Proof</u> of the proposition (existence of the conditional expectation):

Let $P_\mathcal{A} : L^2(X, \mathcal{B}, \mu) \longrightarrow L^2(X, \mathcal{A}, \mu)$ denote the orthogonal projection. For any $A \in \mathcal{A}$, we have $\mathbb{1}_A \in L^2(X, \mathcal{A}, \mu)$, thus for all $f \in L^2(X, \mathcal{B}, \mu)$

$$\int_A P_\mathcal{A} f \, d\mu = \int_A f \, d\mu.$$

We prove that $P_\mathcal{A}$ is continuous with respect to the restriction of the L^1-topology to $L^2(X, \mathcal{B}, \mu)$. Since $P_\mathcal{A}$ is linear, it suffices to show that there is $C > 0$ such that

$$\forall f \in L^2(X, \mathcal{B}, \mu) \quad \|P_\mathcal{A} f\|_1 \le C\|f\|_1.$$

Suppose first that $f \in L^2_\mathbb{R}(X, \mathcal{B}, \mu)$, then $P_\mathcal{A} f \in L^2_\mathbb{R}(X, \mathcal{B}, \mu)$ by the argument marked "not needed" below and

$$\|P_\mathcal{A} f\|_1 = \int_X |P_\mathcal{A} f| \, d\mu = \int_{\{P_\mathcal{A} f > 0\}} |P_\mathcal{A} f| \, d\mu + \int_{\{P_\mathcal{A} f \le 0\}} |P_\mathcal{A} f| \, d\mu$$

$$= \int_{\{P_\mathcal{A} f > 0\}} P_\mathcal{A} f \, d\mu - \int_{\{P_\mathcal{A} f \le 0\}} P_\mathcal{A} f \, d\mu \overset{P_\mathcal{A} f \text{ is } \mathcal{A}\text{-measurable}}{=} \int_{\{P_\mathcal{A} f > 0\}} f \, d\mu - \int_{\{P_\mathcal{A} f \le 0\}} f \, d\mu \le \int_X |f| \, d\mu = \|f\|_1.$$

It follows similarly that

$$\|P_\mathcal{A}(\mathrm{Re} f)\|_1 \le \|f\|_1, \quad \|P_\mathcal{A}(\mathrm{Im} f)\|_1 \le \|f\|_1,$$

and hence the claim follows with $C = 2$.

Given $f \in L^1(X, \mathcal{B}, \mu)$, let $(f_n)_{n \in \mathbb{N}} \in L^2(X, \mathcal{B}, \mu)$ such that $\|f - f_n\|_1 \xrightarrow{n \to \infty} 0$ and define

$$\mathbb{E}(f|\mathcal{A}) = \lim_{n \to \infty} P_\mathcal{A}(f_n) \quad (\text{in } L^1(X, \mathcal{B}, \mu)).$$

Since $L^1(X, \mathcal{A}, \mu) \subseteq L^1(X, \mathcal{B}, \mu)$ is closed, $\mathbb{E}(f|\mathcal{A})$ is $\mathcal{A}$-measurable. Let $A \in \mathcal{A}$ arbitrary and define $\lambda_A : L^1(X, \mathcal{B}, \mu) \longrightarrow \mathbb{C}$,

$$f \longmapsto \lambda_A(f) = \int_A f \, d\mu.$$

Then λ_A is linear and

$$\forall f \in L^1(X, \mathcal{B}, \mu) \quad |\lambda_A(f)| \le \lambda_A(|f|) \le \|f\|_1,$$

hence λ_A is continuous and, thus

$$\int_A \mathbb{E}(f|\mathcal{A}) \, d\mu = \lambda_A(\mathbb{E}(f|\mathcal{A})) = \lim_{n \to \infty} \lambda_A(P_\mathcal{A} f_n) = \lim_{n \to \infty} \lambda_A(f_n) = \lambda_A(f) = \int_A f \, d\mu.$$

It remains to show that for any $f \in L^1(X, \mathcal{B}, \mu)$, the conditional expectation $\mathbb{E}(f|\mathcal{A})$ is uniquely determined by the properties (*) and (**). To this end we first want to assume that w.l.o.g. f and $\mathbb{E}(f|\mathcal{A})$ are real-valued μ-a.s., i.e.,

$$\mathbb{E}(\mathrm{Re}(f)|\mathcal{A}) = \mathrm{Re}(\mathbb{E}(f|\mathcal{A})) \quad \text{and} \quad \mathbb{E}(\mathrm{Im}(f)|\mathcal{A}) = \mathrm{Im}(\mathbb{E}(f|\mathcal{A})) \quad \mu\text{-a.s.}$$

Using the linearity of $\mathbb{E}(\cdot|\mathcal{A})$, it suffices to show that

$$\mathbb{E}(\overline{f}|\mathcal{A}) = \overline{\mathbb{E}(f|\mathcal{A})} \quad \mu\text{-a.s.}$$

Not needed

NOT NEEDED

To this end we argue as follows: Let $A=\{x\in X: \mathrm{Re}(\mathbb{E}(\bar{f}|\mathcal{A}))<\mathrm{Re}(\overline{\mathbb{E}(f|\mathcal{A})})\}$ and note that $A\in\mathcal{A}$. Then

$$\int_A \left(\mathrm{Re}(\mathbb{E}(\bar{f}|\mathcal{A}))-\mathrm{Re}(\overline{\mathbb{E}(f|\mathcal{A})})\right)d\mu=\mathrm{Re}\left(\int_A \mathbb{E}(\bar{f}|\mathcal{A})d\mu\right)-\mathrm{Re}\left(\int_A \overline{\mathbb{E}(f|\mathcal{A})}d\mu\right)$$

$$=\mathrm{Re}\left(\int_A \mathbb{E}(\bar{f}|\mathcal{A})d\mu-\overline{\int_A \mathbb{E}(f|\mathcal{A})d\mu}\right)$$

$$=\mathrm{Re}\left(\int_A \bar{f}d\mu-\overline{\int_A f d\mu}\right)=0,$$

which by definition of A shows $\mu(A)=0$. Similarly one shows that

$$\mathrm{Im}(\mathbb{E}(\bar{f}|\mathcal{A}))=\mathrm{Im}(\overline{\mathbb{E}(f|\mathcal{A})})\quad \mu\text{-a.e.},$$

hence $\mathbb{E}(\bar{f}|\mathcal{A})=\overline{\mathbb{E}(f|\mathcal{A})}$ and thus

$$\mathrm{Re}(\mathbb{E}(f|\mathcal{A}))=\mathbb{E}(\mathrm{Re}(f)|\mathcal{A}) \text{ and } \mathrm{Im}(\mathbb{E}(f|\mathcal{A}))=\mathbb{E}(\mathrm{Im}(f)|\mathcal{A})\quad \mu\text{-a.s.}$$

So suppose that f is $\mathbb{R}$-valued and suppose that $\varphi\in L^1(X,\mathcal{A},\mu)$ satisfies

$$\forall A\in\mathcal{A}\qquad \int_A \varphi d\mu=\int_A f d\mu.$$

Then

$$\mathrm{Im}\int_{\{\mathrm{Im}\varphi>0\}}\varphi\, d\mu=\mathrm{Im}\int_{\{\mathrm{Im}\varphi>0\}} f\, d\mu=0=\mathrm{Im}\int_{\{\mathrm{Im}\varphi<0\}} f\, d\mu=\mathrm{Im}\int_{\{\mathrm{Im}\varphi<0\}}\varphi$$

shows that $\varphi\in L^1_{\mathbb{R}}(X,\mathcal{A},\mu)$. In particular $\mathbb{E}(f|\mathcal{A})\in L^1_{\mathbb{R}}(X,\mathcal{A},\mu)$. Now suppose $g\in L^1_{\mathbb{R}}(X,\mathcal{A},\mu)$ satisfies (***)

$$\int_{\{g>\mathbb{E}(f|\mathcal{A})\}} g\, d\mu=\int_{\{g>\mathbb{E}(f|\mathcal{A})\}} f\, d\mu=\int_{\{g>\mathbb{E}(f|\mathcal{A})\}}\mathbb{E}(f|\mathcal{A})d\mu$$

implies that $\mu(\{g>\mathbb{E}(f|\mathcal{A})\})=0$ and similarly $\mu(\{g<\mathbb{E}(f|\mathcal{A})\})=0$.

The preceding argument implies more generally that

$$\mathbb{E}(\mathrm{Re}(f)|\mathcal{A})=\mathrm{Re}(\mathbb{E}(f|\mathcal{A})) \text{ and } \mathbb{E}(\mathrm{Im}(f)|\mathcal{A})=\mathrm{Im}(\mathbb{E}(f|\mathcal{A})),$$

hence the claim follows by linearity. ∎

End of lecture 14

Theorem (Birkhoff's pointwise/individual ergodic theorem)

Let $(X,\mathcal{B},\mu,T)$ a pmps. For every $f\in L^1(X,\mathcal{B},\mu)$ we have that

$$A_N f=\frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n \xrightarrow{n\to\infty} \mathbb{E}(f|\mathcal{E}_T) \quad \text{“}\int_X f d\mu \text{ if } (X,\mathcal{B},\mu,T) \text{ is ergodic”}$$

in $L^1(X,\mathcal{B},\mu)$ and pointwise μ-a.s.

Lemma

Let $(X,\mathcal{B},\mu,T)$ a pmps and $f\in L^0(X,\mathcal{B},\mu)$. Then

$$f\in L^0(X,\mathcal{E}_{T,\mu},\mu)\iff f=U_T f\quad \mu\text{-a.s.}$$

Proof: We assume w.l.o.g. that $f\in L^0_{\mathbb{R}}(X,\mathcal{B},\mu)$. Given $t\in\mathbb{R}$, let

$$A_t=\{x\in X: f(x)<t\}.$$

Suppose $f\in L^0(X,\mathcal{E}_{T,\mu},\mu)$, then $\forall t\in\mathbb{R}$ $A_t\in\mathcal{E}_{T,\mu}$.

We denote for $k\in\mathbb{Z}$ and $n\in\mathbb{N}$

$$B_{k,n}=A_{\frac{k+1}{n}}-A_{\frac{k}{n}}.$$

If $U_T f\neq f$, then $\exists E\in\mathcal{B}$ s.t. $\mu(E)>0$ and $\forall x\in E$ $U_T f(x)\neq f(x)$

$\Rightarrow \exists n\in\mathbb{N}$ $\mu(\{|U_T f-f|>\frac{1}{n}\})>0$

$\Rightarrow \exists k\in\mathbb{Z}$ $\mu(B_{k,n}\Delta T^{-1}B_{k,n})>0$

$\Rightarrow B_{k,n}\notin\mathcal{E}_{T,\mu}\Rightarrow \exists t\in\mathbb{R}$ $\mu(A_t\Delta T^{-1}A_t)>0$, since

$$B_{k,n}\Delta T^{-1}B_{k,n}\subseteq(A_{\frac{k}{n}}\Delta T^{-1}A_{\frac{k}{n}})\cup(A_{\frac{k+1}{n}}\Delta T^{-1}A_{\frac{k+1}{n}})$$

[Let $E_1,E_2,F_1,F_2\subseteq X$, then

$$(E_1-E_2)\Delta(F_1-F_2)\subseteq(E_1\Delta F_1)\cup(E_2\Delta F_2).$$

Suppose $x\in E_1-E_2 \wedge x\notin F_1-F_2$, then either $x\notin F_1$ or $x\in F_1\cap F_2\subseteq F_2$.

If $x\notin F_1$, then $x\in E_1\Delta F_1$.

If $x\in F_1\cap F_2$, since $x\in E_1-E_2$, we get $x\in E_2\Delta F_2$.]

Hence $A_t\notin\mathcal{E}_{T,\mu}\Rightarrow f\notin L^0(X,\mathcal{E}_{T,\mu},\mu)$.

If $U_T f=f$, then $A_t\in\mathcal{E}_{T,\mu}$ for all $t\in\mathbb{R}$. Thus $f^{-1}((a,b))\in\mathcal{E}_{T,\mu}$ for all $a<b$, since $f^{-1}((a,b))=\bigcup_{n\in\mathbb{N}}(A_b-A_{a+\frac{1}{n}})$. ∎