It remains to prove L^1-convergence. Let $\varphi \in L^2(X,\mathcal{B},\mu)$ such that

$$\|f-\varphi\|_1 < \varepsilon.$$

Then

$$\|A_N f - \mathbb{E}(f|\mathcal{E}_{T,\mu})\|_1 \le \|A_N f - A_N\varphi\|_1 + \|A_N\varphi - \mathbb{E}(\varphi|\mathcal{E}_{T,\mu})\|_1 + \|\mathbb{E}(\varphi - f|\mathcal{E}_{T,\mu})\|_1$$

$$\le \underbrace{3\|f-\varphi\|_1}_{<\varepsilon} + \underbrace{\|A_N\varphi - \mathbb{E}(\varphi|\mathcal{E}_{T,\mu})\|_2}_{\xrightarrow{N\to\infty} 0}$$

$\boxtimes$

Corollary

Let $(X,\mathcal{B},\mu,T)$ ergodic. Let $f\in L^1(X,\mathcal{B},\mu)$. Then

$$\frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n \xrightarrow{N\to\infty} \int_X f\,d\mu \qquad \mu\text{-a.s.}$$

Three examples of ergodic transformations (more to come)

(i) Let $\alpha\in\mathbb{R}-\mathbb{Q}$ and let $R_\alpha:\mathbb{T}\longrightarrow\mathbb{T}$ the rotation by α. We will show that $(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathrm{Leb},R_\alpha)$ is ergodic.

Suppose $f\in L^2(\mathbb{T},\mathrm{Leb})$ and $f\circ R_\alpha = f$. We know from analysis IV that

$$f=\sum_{n\in\mathbb{Z}} a_n(f)e_n,$$

where $e_n:\mathbb{T}\to\mathbb{T}$, $x\mapsto e^{2\pi i n x}$ and $a_n(f)=\int_{\mathbb{T}} f(x)\overline{e_n(x)}\,dx$. We also know that f is completely and uniquely determined by $(a_n(f))_{n\in\mathbb{Z}}$ and that $\sum_{n\in\mathbb{Z}}|a_n(f)|^2=\|f\|_2^2$.

Let $n\in\mathbb{Z}$ arbitrary, then

$$a_n(f)=\int_{\mathbb{T}} f(x)\overline{e_n(x)}\,dx = \int_{\mathbb{T}}(f\circ R_\alpha)(x)\overline{e_n(x)}\,dx$$

$$=\int_{\mathbb{T}} f(x+\alpha)\overline{e_n(x)}\,dx = \int_{\mathbb{T}} f(x)\overline{e_n(x-\alpha)}\,dx = e_n(\alpha)\int_{\mathbb{T}} f(x)\overline{e_n(x)}\,dx$$

$$=e^{2\pi i n\alpha}a_n(f).$$

Hence, since $\alpha\in\mathbb{R}\setminus\mathbb{Q} \Rightarrow \forall n\in\mathbb{Z}-\{0\}$ $e^{2\pi i n\alpha}\neq 1$, it follows that $a_n(f)=0$ for all $n\in\mathbb{Z}\setminus\{0\}$, i.e.,

$$f=a_0(f)\mathbb{1}_{\mathbb{T}}.$$

(ii) Again, let $f\in L^2(\mathbb{T},\mathrm{Leb})$ and suppose $f\circ T_p=f$. Let $n\in\mathbb{Z}$ and $k\in\mathbb{N}$, then

$$a_{p^k n}(f)=\int_{\mathbb{T}} f(x)\overline{e_{p^k n}(x)}\,dx = \int_{\mathbb{T}} f(x)\overline{e_n(p^k x)}\,dx$$

$$=\int_{\mathbb{T}}(f\circ T_p^k)(x)\overline{e_n(T_p^k x)}\,dx = \int_{\mathbb{T}} f(x)\overline{e_n(x)}\,dT^k_{p*}\mathrm{Leb}(x)$$

$$=\int_{\mathbb{T}} f(x)\overline{e_n(x)}\,dx = a_n(f).$$

If $n\in\mathbb{Z}-\{0\}$, $k\mapsto p^k n$ is injective and, thus,

$$\infty > \|f\|_2^2 = \sum_{m\in\mathbb{Z}}|a_m(f)|^2 \ge \sum_{k\in\mathbb{N}}|a_{p^k n}(f)|^2 = \sum_{k\in\mathbb{N}}|a_n(f)|^2$$

implies $a_n(f)=0$.

(iii) Let $A=\{0,\dots,p-1\}$, $X=A^{\mathbb{Z}}$, $\mathcal{B}=\mathcal{B}(X)$, $\mu=\mathbb{P}^{\otimes\mathbb{Z}}$ ($\mathbb{P}$ prob), $\sigma:X\to X$ left-shift. Then $(X,\mathcal{B},\mu,\sigma)$ is ergodic.

Recall

An algebra $\mathcal{A}$ over X is a subset $\mathcal{A}\subseteq 2^X$ which is a semialgebra such that for all $A\in\mathcal{A}$ we have $X-A\in\mathcal{A}$.

Theorem

Let $(X,\mathcal{B},\mu)$ a probability space and $\mathcal{A}\subseteq\mathcal{B}$ an algebra, then

$$\{B\in\mathcal{B}: \forall\varepsilon>0\ \exists A\in\mathcal{A}\ \mu(A\Delta B)<\varepsilon\}$$

is a σ-algebra.

In particular, $\sigma(\mathcal{A})\subseteq\{B\in\mathcal{B}:\forall\varepsilon>0\ \exists A\in\mathcal{A}\ \mu(A\Delta B)<\varepsilon\}$.

We let $\mathcal{A}$ be the algebra generated by the cylinder sets. Any element in $\mathcal{A}$ is a finite union of cylinders.

Let $E\in\mathcal{B}$ invariant, i.e., $T^{-1}E=E$. Since $\mathcal{B}$ is generated by $\mathcal{A}$, let $\varepsilon>0$ arbi trary and $A \in \mathcal{A}$ such that $\mu(E \Delta A) < \varepsilon$. Then

$$|\mu(E) - \mu(A)| = |\mu(E \cap A) + \mu(E \setminus A) - \mu(A \cap E) - \mu(A \setminus E)|$$

$$\leq \mu(E \setminus A) + \mu(A \setminus E) = \mu(A \Delta E) < \varepsilon. \qquad (*)$$

Choose $n_0 \in \mathbb{N}$ large such that the set of indices determining $\sigma^{-n_0}(A) = B$ is disjoint from the indices determining A, so that

$$\mu(A \cap B) = \mu(A)\mu(B) = \mu(A)^2.$$

Then

$$\mu(E \Delta B) = \mu(T^{-n_0} E \Delta B) = \mu(E \Delta A) < \varepsilon.$$

Since $E \Delta (A \cap B) \subseteq (E \Delta A) \cup (E \Delta B)$, we have $\mu(E \Delta (A \cap B)) < 2\varepsilon$. Hence by the same argument as above $(*)$

$$|\mu(E) - \mu(A \cap B)| < 2\varepsilon$$

and

$$|\mu(E) - \mu(E)^2| \leq |\mu(E) - \mu(A \cap B)| + |\mu(A)^2 - \mu(E)^2|$$

$$< 2\varepsilon + \mu(A)|\mu(A) - \mu(E)| + \mu(E)|\mu(A) - \mu(E)| < 4\varepsilon.$$

But ε was arbitrary, thus $\mu(E) \in \{0, 1\}$. $\square$

Remark: - $(A^{\mathbb{N}}, \mathcal{B}(A^{\mathbb{N}}), p^{\otimes \mathbb{N}}, \sigma)$ is ergodic

a) because $(x_n)_{n \in \mathbb{Z}} \xmapsto{\pi} (x_n)_{n \in \mathbb{N}}$ is a factor map. Indeed, if $f : A^{\mathbb{N}} \to \mathbb{C}$ is a σ-invariant map, then

$$f \circ \pi = f \circ \underset{\substack{\uparrow \\ \text{one-sided}}}{\sigma} \circ \pi = f \circ \pi \circ \underset{\substack{\uparrow \\ \text{two-sided}}}{\sigma}$$

is a σ-invariant function $f \circ \pi : A^{\mathbb{Z}} \to \mathbb{C}$. Hence $f \circ \pi$ is constant and thus so is f (almost everywhere),

b) if p is uniform, because

$$(A^{\mathbb{N}}, \mathcal{B}(A^{\mathbb{N}}), p^{\otimes \mathbb{N}}, \sigma) \cong (\mathbb{T}, \mathcal{B}(\mathbb{T}), Leb, T_p).$$

For non-uniform p, this yields additional measures which are T_p-invariant and ergodic! So $(\mathbb{T}, \mathcal{B}(\mathbb{T}), Leb, T_p)$ admits at least as many invariant and ergodic probability measures as there are probability vectors on p symbols.

by argument a)

Corollary

Let $\alpha \in \mathbb{R} - \mathbb{Q}$, $f \in C(\mathbb{T})$, then for a.e. $x \in \mathbb{T}$

$$\frac{1}{N} \sum_{n=0}^{N-1} f(x + n\alpha) \xrightarrow{N \to \infty} \int_{\mathbb{T}} f(x)\,dx.$$

irrelevant!

relevant!

Let $p \in \mathbb{N} - \{0\}$, $f \in C(\mathbb{T})$, then for a.e. $x \in \mathbb{T}$

$$\frac{1}{N} \sum_{n=0}^{N-1} f(p^n x) \xrightarrow{N \to \infty} \int_{\mathbb{T}} f(x)\,dx$$

Proof that the convergence for the irrational rotation is valid for every $x \in \mathbb{T}$:

Let $m \in \mathbb{Z}$, then

$$\frac{1}{N} \sum_{n=0}^{N-1} e_m(x + n\alpha) = \frac{e_m(x)}{N} \sum_{n=0}^{N-1} e_m(n\alpha) = \begin{cases} e_m(x) = 1 & \text{if } m = 0, \\ \frac{1}{N} \frac{e_m(N\alpha) - 1}{e_m(\alpha) - 1} & \text{if } m \neq 0. \end{cases}$$

So $\lim_{N \to \infty} A_N e_m(x) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{else}. \end{cases}$ Since $\int_{\mathbb{T}} e_m(x)\,dx = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{else}, \end{cases}$

the claim follows for finite linear combinations of $\{e_m : m \in \mathbb{Z}\}$. These "trigonometric polynomials" are dense in $C(\mathbb{T})$.