NOT NEEDED

To this end we argue as follows: Let $A=\{x\in X: \mathrm{Re}(\mathbb{E}(\bar f|\mathcal{A}))<\mathrm{Re}(\overline{\mathbb{E}(f|\mathcal{A})})\}$ and note that $A\in\mathcal{A}$. Then

$$\int_A \left(\mathrm{Re}(\mathbb{E}(\bar f|\mathcal{A}))-\mathrm{Re}(\overline{\mathbb{E}(f|\mathcal{A})})\right)d\mu=\mathrm{Re}\left(\int_A\mathbb{E}(\bar f|\mathcal{A})d\mu\right)-\mathrm{Re}\left(\int_A\overline{\mathbb{E}(f|\mathcal{A})}d\mu\right)$$

$$=\mathrm{Re}\left(\int_A\mathbb{E}(\bar f|\mathcal{A})d\mu-\overline{\int_A\mathbb{E}(f|\mathcal{A})d\mu}\right)$$

$$=\mathrm{Re}\left(\int_A\bar f d\mu-\overline{\int_A f d\mu}\right)=0,$$

which by definition of A shows $\mu(A)=0$. Similarly one shows that

$$\mathrm{Im}(\mathbb{E}(\bar f|\mathcal{A}))=\mathrm{Im}(\overline{\mathbb{E}(f|\mathcal{A})})\quad \mu\text{-e.},$$

hence $\mathbb{E}(\bar f|\mathcal{A})=\overline{\mathbb{E}(f|\mathcal{A})}$ and thus

$$\mathrm{Re}(\mathbb{E}(f|\mathcal{A}))=\mathbb{E}(\mathrm{Re}(f)|\mathcal{A})\text{ and }\mathrm{Im}(\mathbb{E}(f|\mathcal{A}))=\mathbb{E}(\mathrm{Im}(f)|\mathcal{A})\ \mu\text{-a.s.}$$

So suppose that f is $\mathbb{R}$-valued and suppose that $\varphi\in L^1(X,\mathcal{A},\mu)$ satisfies

$$\forall A\in\mathcal{A}\qquad \int_A\varphi d\mu=\int_A f d\mu.$$

Then

$$\mathrm{Im}\int_{\{\mathrm{Im}\varphi>0\}}\varphi\, d\mu=\mathrm{Im}\int_{\{\mathrm{Im}\varphi>0\}}f\, d\mu=0=\mathrm{Im}\int_{\{\mathrm{Im}\varphi<0\}}f\,d\mu=\mathrm{Im}\int_{\{\mathrm{Im}\varphi<0\}}\varphi$$

shows that $\varphi\in L^1_{\mathbb{R}}(X,\mathcal{A},\mu)$. In particular $\mathbb{E}(f|\mathcal{A})\in L^1_{\mathbb{R}}(X,\mathcal{A},\mu)$. Now suppose $g\in L^1_{\mathbb{R}}(X,\mathcal{A},\mu)$ satisfies (**)

$$\int_{\{g>\mathbb{E}(f|\mathcal{A})\}}g\,d\mu=\int_{\{g>\mathbb{E}(f|\mathcal{A})\}}f\,d\mu=\int_{\{g>\mathbb{E}(f|\mathcal{A})\}}\mathbb{E}(f|\mathcal{A})d\mu$$

implies that $\mu(\{g>\mathbb{E}(f|\mathcal{A})\})=0$ and similarly $\mu(\{g<\mathbb{E}(f|\mathcal{A})\})=0$.

The preceding argument implies more generally that

$$\mathbb{E}(\mathrm{Re}(f)|\mathcal{A})=\mathrm{Re}(\mathbb{E}(f|\mathcal{A}))\text{ and }\mathbb{E}(\mathrm{Im}(f)|\mathcal{A})=\mathrm{Im}(\mathbb{E}(f|\mathcal{A})),$$

hence the claim follows by linearity. ∎

End of lecture 14

Theorem (Birkhoff's pointwise/individual ergodic theorem)

Let $(X,\mathcal{B},\mu,T)$ a pmps. For every $f\in L^1(X,\mathcal{B},\mu)$ we have that

$$A_N f=\frac{1}{N}\sum_{n=0}^{N-1}f\circ T^n\xrightarrow{n\to\infty}\mathbb{E}(f|\mathcal{E}_T)$$

$\int_X f d\mu$ if $(X,\mathcal{B},\mu,T)$ is ergodic

in $L^1(X,\mathcal{B},\mu)$ and pointwise μ-a.s.

Lemma

Let $(X,\mathcal{B},\mu,T)$ a pmps and $f\in L^0(X,\mathcal{B},\mu)$. Then

$$f\in L^0(X,\mathcal{E}_{T,\mu},\mu)\iff f=U_T f\quad\mu\text{-a.s.}$$

Proof: We assume w.l.o.g. that $f\in L^0_{\mathbb{R}}(X,\mathcal{B},\mu)$. Given $t\in\mathbb{R}$, let

$$A_t=\{x\in X: f(x)<t\}.$$

Suppose $f\in L^0(X,\mathcal{E}_{T,\mu},\mu)$, then $\forall t\in\mathbb{R}$ $A_t\in\mathcal{E}_{T,\mu}$.

We denote for $k\in\mathbb{Z}$ and $n\in\mathbb{N}$

$$B_{k,n}=A_{\frac{k+1}{n}}-A_{\frac{k}{n}}.$$

If $U_Tf\neq f$, then $\exists E\in\mathcal{B}$ s.t. $\mu(E)>0$ and $\forall x\in E$ $U_Tf(x)\neq f(x)$

$\Rightarrow \exists n\in\mathbb{N}$ $\mu(\{|U_Tf-f|>\frac{1}{n}\})>0$

$\Rightarrow \exists k\in\mathbb{Z}$ $\mu(B_{k,n}\Delta T^{-1}B_{k,n})>0$

$\Rightarrow B_{k,n}\notin\mathcal{E}_{T,\mu}\Rightarrow\exists t\in\mathbb{R}$ $\mu(A_t\Delta T^{-1}A_t)>0$, since

$$B_{k,n}\Delta T^{-1}B_{k,n}\subseteq(A_{\frac{k}{n}}\Delta T^{-1}A_{\frac{k}{n}})\cup(A_{\frac{k+1}{n}}\Delta T^{-1}A_{\frac{k+1}{n}})$$

[Let $E_1,E_2,F_1,F_2\subseteq X$, then

$(E_1-E_2)\Delta(F_1-F_2)\subseteq(E_1\Delta F_1)\cup(E_2\Delta F_2)$.

Suppose $x\in E_1-E_2$, $x\notin F_1-F_2$, then either $x\notin F_1$ or $x\in F_1\cap F_2\subseteq F_2$.

If $x\notin F_1$, then $x\in E_1\Delta F_1$.

If $x\in F_1\cap F_2$, since $x\in E_1-E_2$, we get $x\in E_2\Delta F_2$.]

Hence $A_t\notin\mathcal{E}_{T,\mu}\Rightarrow f\notin L^0(X,\mathcal{E}_{T,\mu},\mu)$.

If $U_Tf=f$, then $A_t\in\mathcal{E}_{T,\mu}$ for all $t\in\mathbb{R}$. Thus $f^{-1}((a,b))\in\mathcal{E}_{T,\mu}$ for all $a<b$,

since $f^{-1}((a,b))=\bigcup_{n\in\mathbb{N}}(A_b-A_{a+\frac{1}{n}})$. ∎

Exercise: Let $(X,\mathcal{B},\mu,T)$ a pmps. TFAE:

(i) $(X,\mathcal{B},\mu,T)$ is ergodic.

(ii) $\forall f\in L^2(X,\mathcal{B},\mu)$ $\mathbb{E}(f|\mathcal{E}_{T,\mu})$ is constant.

<u>Proof</u> of Birkhoff's pointwise ergodic theorem

Let $f \in L^1(X,\mathcal{B},\mu)$, then $\mathbb{E}(f|\mathcal{E}_T) = \mathbb{E}(f|\mathcal{E}_{T,\mu})$ μ-a.e. Since $\mathcal{E}_T \subseteq \mathcal{E}_{T,\mu}$, we only have to verify that

$$\forall A \in \mathcal{E}_{T,\mu} \qquad \int_A \mathbb{E}(f|\mathcal{E}_T)\,d\mu = \int_A f\,d\mu.$$

But, as shown earlier, there is $C \in \mathcal{E}_T$ s.t. $\mu(A \Delta C) = 0$, hence

$$\int_A \mathbb{E}(f|\mathcal{E}_T)\,d\mu = \int_X \mathbb{1}_A\, \mathbb{E}(f|\mathcal{E}_T)\,d\mu = \int_X \mathbb{1}_C\, \mathbb{E}(f|\mathcal{E}_T)\,d\mu$$

$$= \int_C \mathbb{E}(f|\mathcal{E}_T)\,d\mu = \int_C f\,d\mu = \int_X \mathbb{1}_C f\,d\mu = \int_X \mathbb{1}_A f\,d\mu = \int_A f\,d\mu.$$

So it suffices to prove $A_N f \longrightarrow \mathbb{E}(f|\mathcal{E}_{T,\mu})$. It will be convenient to denote for $g \in L^1(X,\mathcal{B},\mu)$ $S_0 g = 0$ and $S_N = \sum_{n=0}^{N-1} U_T^n g$.
Let $\varepsilon > 0$ and $g = f - \mathbb{E}(f|\mathcal{E}_{T,\mu}) - \varepsilon$. Then

$$A_N g = A_N f - \mathbb{E}(f|\mathcal{E}_{T,\mu}) - \varepsilon \quad \mu\text{-a.e.}$$

by the preceding lemma. We want to show that $\limsup_N A_N g \le 0$ μ-a.e.
Note that $\limsup_N A_N g(x) > 0 \Rightarrow \limsup_N S_N g(x) = \infty$, hence it suffices to show that

$$\mu(\underbrace{\{x \in X : \sup_N S_N g(x) = \infty\}}_{A}) = 0$$

Note that $U_T S_N g(x) = S_{N+1} g(x) - g(x)$, thus $\mu(A \Delta T^{-1}A) = 0$, i.e., $A \in \mathcal{E}_{T,\mu}$.
(for any fixed [illegible] $x \in A \Leftrightarrow Tx \in A$.)

Given $n \in \mathbb{N}$, let $M_n = \max_{0 \le k \le n} S_k g$. Then $M_{n+1} = \max\{0, g + M_n \circ T\}$

$$\Rightarrow \forall x \in X \quad M_{n+1}(x) - M_n(Tx) = \max\{-M_n(Tx), g(x)\}.$$

For μ-a.e. $x \in A$ we have that $M_n(Tx) \xrightarrow{n \to \infty} \infty$, hence for μ-a.e. $x \in A$

$$\lim_{n \to \infty} M_{n+1}(x) - M_n(Tx) = g(x)$$

$M_n(Tx) = \max\{0, g(Tx), \ldots, [illegible]\} \ge 0$

We also have $g(x) \le \max\{-M_n(Tx), g(x)\} = M_{n+1}(x) - M_n(Tx) \le \max\{0, g(x)\}$, i.e.,

$$|M_{n+1} - M_n \circ T| \le |g| \in L^1(X,\mathcal{B},\mu).$$

Hence we can use Lebesgue's dominated convergence theorem to deduce

$$\lim_{n \to \infty} \int_A M_{n+1} - M_n \circ T\,d\mu = \int_A g\,d\mu.$$

Note that

$$\int_A M_n \circ T\,d\mu = \int_X \mathbb{1}_A \cdot M_n \circ T\,d\mu \overset{A \in \mathcal{E}_{T,\mu}}{=} \int_X (\mathbb{1}_A \cdot M_n) \circ T\,d\mu = \int_X \mathbb{1}_A \cdot M_n\,dT_*\mu$$

$$= \int_X \mathbb{1}_A \cdot M_n\,d\mu = \int_A M_n\,d\mu,$$

Thus

$$\lim_{n \to \infty} \int_A M_{n+1} - M_n \circ T\,d\mu = \lim_{n \to \infty} \int_A M_{n+1} - M_n\,d\mu \ge 0$$

$$\Rightarrow \int_A g\,d\mu \ge 0.$$

Since $A \in \mathcal{E}_{T,\mu}$, we have

$$\int_A g\,d\mu = \int_A (f - \mathbb{E}(f|\mathcal{E}_{T,\mu}) - \varepsilon)\,d\mu = -\varepsilon\mu(A),$$

Thus $\mu(A) = 0$.
It follows that $\limsup_N A_N g \le 0$ μ-a.s., thus — since ε was arbitrary — we find that

$$\limsup_N A_N f \le \mathbb{E}(f|\mathcal{E}_{T,\mu}) \quad \mu\text{-a.s.}$$

Similarly, we have that replacing f by $-f$ in the argument

$$\begin{aligned} \limsup_N A_N(-f) &\le \mathbb{E}(-f|\mathcal{E}_{T,\mu}) = -\mathbb{E}(f|\mathcal{E}_{T,\mu}) \\ &\| \\ -\liminf_N A_N f&, \end{aligned} \qquad \mu\text{-a.s.}$$

Thus $A_N f \longrightarrow \mathbb{E}(f|\mathcal{E}_{T,\mu})$ μ-a.s.

It remains to prove L^1-convergence. Let $\varphi \in L^2(X,\mathcal{B},\mu)$ such that
$$\|f-\varphi\|_1 < \varepsilon.$$

Then
$$\|A_N f - \mathbb{E}(f|\mathcal{E}_{T,\mu})\|_1 \le \|A_N f - A_N\varphi\|_1 + \|A_N\varphi - \mathbb{E}(\varphi|\mathcal{E}_{T,\mu})\|_1 + \|\mathbb{E}(\varphi - f|\mathcal{E}_{T,\mu})\|_1$$
$$\le \underbrace{3\|f-\varphi\|_1}_{<\varepsilon} + \underbrace{\|A_N\varphi - \mathbb{E}(\varphi|\mathcal{E}_{T,\mu})\|_2}_{\xrightarrow{N\to\infty} 0}$$
$\boxtimes$

Corollary

Let $(X,\mathcal{B},\mu,T)$ ergodic. Let $f\in L^1(X,\mathcal{B},\mu)$. Then
$$\frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n \xrightarrow{N\to\infty} \int_X f\,d\mu \qquad \mu\text{-a.s.}$$

Three examples of ergodic transformations (more to come)

(i) Let $\alpha\in\mathbb{R}-\mathbb{Q}$ and let $R_\alpha:\mathbb{T}\to\mathbb{T}$ the rotation by α. We will show that $(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathrm{Leb},R_\alpha)$ is ergodic.

Suppose $f\in L^2(\mathbb{T},\mathrm{Leb})$ and $f\circ R_\alpha = f$. We know from analysis IV that
$$f=\sum_{n\in\mathbb{Z}} a_n(f)e_n,$$
where $e_n:\mathbb{T}\to\mathbb{T}$, $x\mapsto e^{2\pi i n x}$ and $a_n(f)=\int_{\mathbb{T}} f(x)\overline{e_n(x)}\,dx$. We also know that f is completely and uniquely determined by $(a_n(f))_{n\in\mathbb{Z}}$ and that $\sum_{n\in\mathbb{Z}}|a_n(f)|^2=\|f\|_2^2$.

Let $n\in\mathbb{Z}$ arbitrary, then
$$a_n(f)=\int_{\mathbb{T}} f(x)\overline{e_n(x)}\,dx = \int_{\mathbb{T}}(f\circ R_\alpha)(x)\overline{e_n(x)}\,dx$$
$$=\int_{\mathbb{T}} f(x+\alpha)\overline{e_n(x)}\,dx = \int_{\mathbb{T}} f(x)\overline{e_n(x-\alpha)}\,dx = e_n(\alpha)\int_{\mathbb{T}} f(x)\overline{e_n(x)}\,dx$$
$$=e^{2\pi i n\alpha}a_n(f).$$

Hence, since $\alpha\in\mathbb{R}\setminus\mathbb{Q} \Rightarrow \forall n\in\mathbb{Z}-\{0\}\ e^{2\pi i n\alpha}\neq 1$, it follows that $a_n(f)=0$ for all $n\in\mathbb{Z}\setminus\{0\}$, i.e.,
$$f=a_0(f)\mathbb{1}_{\mathbb{T}}.$$

(ii) Again, let $f\in L^2(\mathbb{T},\mathrm{Leb})$ and suppose $f\circ T_p = f$. Let $n\in\mathbb{Z}$ and $k\in\mathbb{N}$, then
$$a_{p^k n}(f)=\int_{\mathbb{T}} f(x)\overline{e_{p^k n}(x)}\,dx = \int_{\mathbb{T}} f(x)\overline{e_n(p^k x)}\,dx$$
$$=\int_{\mathbb{T}}(f\circ T_p^k)(x)\overline{e_n(T_p^k x)}\,dx = \int_{\mathbb{T}} f(x)\overline{e_n(x)}\,d\,T^k_{p*}\mathrm{Leb}(x)$$
$$=\int_{\mathbb{T}} f(x)\overline{e_n(x)}\,dx = a_n(f).$$

If $n\in\mathbb{Z}-\{0\}$, $k\mapsto p^k n$ is injective and, thus,
$$\infty > \|f\|_2^2 = \sum_{m\in\mathbb{Z}}|a_m(f)|^2 \ge \sum_{k\in\mathbb{N}}|a_{p^k n}(f)|^2 = \sum_{k\in\mathbb{N}}|a_n(f)|^2$$
implies $a_n(f)=0$.

(iii) Let $A=\{0,\dots,p-1\}$, $X=A^{\mathbb{Z}}$, $\mathcal{B}=\mathcal{B}(X)$, $\mu=p^{\otimes\mathbb{Z}}$ (p prob), $\sigma:X\to X$ left-shift. Then $(X,\mathcal{B},\mu,\sigma)$ is ergodic.

Recall

An algebra $\mathcal{A}$ over X is a subset $\mathcal{A}\subseteq 2^X$ which is a semialgebra such that for all $A\in\mathcal{A}$ we have $X-A\in\mathcal{A}$.

Theorem

Let $(X,\mathcal{B},\mu)$ a probability space and $\mathcal{A}\subseteq\mathcal{B}$ an algebra, then
$$\{B\in\mathcal{B}:\forall\varepsilon>0\ \exists A\in\mathcal{A}\ \mu(A\Delta B)<\varepsilon\}$$
is a σ-algebra.

In particular, $\sigma(\mathcal{A})\subseteq\{B\in\mathcal{B}:\forall\varepsilon>0\ \exists A\in\mathcal{A}\ \mu(A\Delta B)<\varepsilon\}$.

We let $\mathcal{A}$ be the algebra generated by the cylinder sets. Any element in $\mathcal{A}$ is a finite union of cylinders.

Let $E\in\mathcal{B}$ invariant, i.e., $T^{-1}E=E$. Since $\mathcal{B}$ is generated by $\mathcal{A}$, let $\varepsilon>0$ arbi