Proof: (i) If $n=0$, this is part of the definition of a (p,P)-Markov chain, $p=(p_s)_{s\in S}$.
Let $n\geq 1$, then

$$\mu(\pi_n=s)=\sum_{t\in S}\mu(\pi_n=s,\pi_{n-1}=t)$$

$$=\sum_{t\in S}\mu(\pi_n=s\,|\,\pi_{n-1}=t)\,\mu(\pi_{n-1}=t)$$

and, by induction, letting $t_n:=s$

$$\mu(\pi_n=s)=\sum_{(t_0,\dots,t_{n-1})\in S^n}\mu(\pi_0=t_0)\prod_{k=0}^{n-1}\mu(\pi_{k+1}=t_{k+1}\,|\,\pi_k=t_k)$$

$$=\sum_{(t_0,\dots,t_{n-1})\in S^n}p_{t_0}\prod_{k=0}^{n-1}p_{t_kt_{k+1}}=(*)$$

One verifies that $(*)=(pP^n)_s$ by induction.

(ii) $$\mu(\pi_{n+m}=t\,|\,\pi_m=s)=\frac{\mu(\Sigma_{n+m}(t)\cap\Sigma_m(s))}{\mu(\Sigma_m(s))}\underset{\text{shift-invariance}}{\overset{\uparrow}{=}}\frac{\mu(\Sigma_n(t)\cap\mathcal{C}(s))}{\mu(\mathcal{C}(s))}=\mu(\pi_n=t\,|\,\pi_0=s)$$

Note that by induction, for $(s_0,\dots,s_n)\in S^{n+1}$

$$\mu(\mathcal{C}(s_0,\dots,s_n))=\mu(\pi_0=s_0,\dots,\pi_n=s_n)=\mu(\pi_n=s_n\,|\,\pi_0=s_0,\dots,\pi_{n-1}=s_{n-1})\cdot\mu(\pi_{n-1}=s_{n-1},\dots,\pi_0=s_0)$$

$$\overset{\text{Markov assumption}}{=}\mu(\pi_n=s_n\,|\,\pi_{n-1}=s_{n-1})\cdot\mu(\pi_{n-1}=s_{n-1},\dots,\pi_0=s_0)$$

$$\overset{\text{induction}}{=}\mu(\pi_0=s_0)\prod_{k=0}^{n-1}\mu(\pi_{k+1}=s_{k+1}\,|\,\pi_k=s_k)$$

$$=p_{s_0}\prod_{k=0}^{n-1}p_{s_ks_{k+1}}.$$

Hence, letting $t=:s_n,s=:s_0$, we have

$$\mu(\Sigma_n(t)\cap\mathcal{C}(s))=\sum_{(s_1,\dots,s_{n-1})\in S^{n-1}}\mu(\mathcal{C}(s_0,\dots,s_n))$$

$$=\sum_{(s_1,\dots,s_{n-1})\in S^{n-1}}p_{s_0}\prod_{k=0}^{n-1}p_{s_ks_{k+1}},$$

i.e.,

$$\mu(\pi_{n+m}=t\,|\,\pi_m=s)=\sum_{(s_1,\dots,s_{n-1})\in S^{n-1}}\prod_{k=0}^{n-1}p_{s_ks_{k+1}}=(P^n)_{s_0s_n}=(P^n)_{st}$$ ☐

Corollary:
Let $(S^{\mathbb{N}_0},\mathcal{B}(S^{\mathbb{N}_0}),\mu)$ be a (p,P)-Markov chain. Then $(S^{\mathbb{N}_0},\mathcal{B}(S^{\mathbb{N}_0}),\mu,\sigma)$ is a pmps.

Proof: We have shown that the map

$$(s_0,\dots,s_n)\longmapsto p_{s_0}\prod_{k=0}^{n-1}p_{s_ks_{k+1}}$$

defines a shift-invariant measure on the semialgebra generated by the cylinder sets. ☐

Proposition ("Markov property" for stationary Markov chains)
Let $(S^{\mathbb{N}_0},\mathcal{B}(S^{\mathbb{N}_0}),\mu)$ be a (p,P)-Markov chain. Let $f\in L^1(S^{\mathbb{N}_0},\mu)$ and $n\in\mathbb{N}_0$.
Then

$$\mathbb{E}(f\circ\sigma^n\,|\,\mathcal{F}_n)=\mathbb{E}(f\,|\,\mathcal{B}_0)\circ\sigma^n\quad\mu\text{-a.s.}$$

$\mathbb{E}(f\,|\,\mathcal{B}_0)$ is a linear combination of $\{\mathbb{1}_{\mathcal{C}(s)}:s\in S\}$ and hence only depends on the 0-th coordinate.

Proof: Let $\underline{x}=(x_\ell)_{\ell\in\mathbb{N}_0}\in S^{\mathbb{N}_0}$, then

$$\mathbb{E}(f\circ\sigma^n\,|\,\mathcal{F}_n)(\underline{x})\overset{\text{Markov}}{=}\mathbb{E}(f\circ\sigma^n\,|\,\mathcal{B}_n)(\underline{x})=\frac{\mathbb{E}((f\circ\sigma^n)\mathbb{1}_{\Sigma_n(x_n)})}{\mu(\Sigma_n(x_n))}$$

$\Sigma_n(x_n)=\sigma^{-n}(\mathcal{C}(x_n))$

$$=\frac{\mathbb{E}((f\circ\sigma^n)(\mathbb{1}_{\mathcal{C}(x_n)}\circ\sigma^n))}{\mu(\sigma^{-n}(\mathcal{C}(x_n)))}\underset{\text{invariance}}{=}\frac{\mathbb{E}(f\cdot\mathbb{1}_{\mathcal{C}(x_n)})}{\mu(\mathcal{C}(x_n))}$$

$$=\sum_{s\in S}\frac{\mathbb{E}(f\cdot\mathbb{1}_{\mathcal{C}(s)})}{\mu(\mathcal{C}(s))}(\mathbb{1}_{\mathcal{C}(s)}\circ\sigma^n)(\underline{x})=\mathbb{E}(f\,|\,\mathcal{B}_0)\circ\sigma^n$$ ☐

Definition (Stopping time)

A map $\tau : (S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0})) \longrightarrow (\mathbb{N}_0, 2^{\mathbb{N}_0})$ is a stopping time if

$$\forall n \in \mathbb{N}_0 \quad \{\tau \leq n\} \in \mathcal{F}_n .$$

The σ-algebra

$$\mathcal{F}_\tau = \{A \in \mathcal{B}(S^{\mathbb{N}_0}) : \forall n \in \mathbb{N}_0 \ A \cap \{\tau \leq n\} \in \mathcal{F}_n\}$$

is called the stopping time σ-algebra.

Theorem ("Strong Markov property" for stationary Markov chains)

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be a (p,P)-Markov chain. Let $f \in L^1(S^{\mathbb{N}_0}, \mu)$ and let $\tau : (S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0})) \longrightarrow (\mathbb{N}_0, 2^{\mathbb{N}_0})$ be a stopping time. Then

$$\mathbb{1}_{\{\tau<\infty\}} \mathbb{E}(f \circ \sigma^\tau \mid \mathcal{F}_\tau) = \mathbb{1}_{\{\tau<\infty\}} (\mathbb{E}(f \mid \mathcal{B}_0) \circ \sigma^\tau) \quad \mu\text{-a.s.}$$

Proof: Let $F \in \mathcal{F}_\tau$, then

$$\int_{F\cap\{\tau<\infty\}} f \circ \sigma^\tau \, d\mu = \sum_{n\in\mathbb{N}_0} \int_{\underbrace{F\cap\{\tau=n\}}_{\in \mathcal{F}_n}} f \circ \sigma^n \, d\mu = \sum_{n \in \mathbb{N}_0} \int_{F\cap\{\tau=n\}} \mathbb{E}(f \mid \mathcal{B}_0) \circ \sigma^n \, d\mu$$

$$= \int_{F\cap\{\tau<\infty\}} \mathbb{E}(f \mid \mathcal{B}_0) \circ \sigma^\tau \, d\mu$$

$\blacksquare$

Recall once again: If $f \in L^1(\Omega, \mathcal{F}, \mu)$ for a probability space $(\Omega, \mathcal{F}, \mu)$ and if $\mathcal{P} = \{P_0, P_1, \ldots\}$ is a measurable partition of Ω, then

$$\mathbb{E}(f \mid \sigma(\mathcal{P})) = \sum_{i=0}^{\infty} \frac{\mathbb{E}(f \mathbb{1}_{P_i})}{\mu(P_i)} \mathbb{1}_{P_i} .$$

It is customary, to define

$$\mathbb{E}(f \mid P_i) := \frac{\mathbb{E}(f \mathbb{1}_{P_i})}{\mu(P_i)} ,$$

and for $A \in \mathcal{F}$ one writes

$$\mu(A \mid P_i) = \mathbb{E}(\mathbb{1}_A \mid P_i) = \frac{\mu(A \cap P_i)}{\mu(P_i)} .$$

For example, $\mu(A \mid \pi_0 = s) = \dfrac{\mu(A \cap E(s))}{\mu(E(s))}$

Definition

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be a (p,P)-Markov chain, let $s \in S$. Then

(i) s is recurrent if $\mu(\limsup_n \{\pi_n = s\} \mid \pi_0 = s) = 1$, and

(ii) s is transient if $\mu(\limsup_n \{\pi_n = s\} \mid \pi_0 = s) = 0$.

Definition

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be a (p,P)-Markov chain, let $s \in S$. The "first hitting time" for s is the stopping time $\tau_s = \inf\{n \in \mathbb{N} : \pi_n = s\}$

Theorem

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be a (p,P)-Markov chain, let $s \in S$. Then

(i) if $\mu(\tau_s < \infty \mid \pi_0 = s) = 1$, then s is recurrent and $\sum_{n=0}^{\infty} p_{ss}^{(n)} = \infty$, and

(ii) if $\mu(\tau_s < \infty \mid \pi_0 = s) < 1$, then s is transient and $\sum_{n=0}^{\infty} p_{ss}^{(n)} < \infty$.

In particular, every state is either transient or recurrent.

We introduce some notation: Let $s \in S$, then

$$\tau_s^{(0)} = 0, \quad \tau_s^{(1)} = \tau_s, \quad \text{and for } r \in \mathbb{N}-\{1\} \quad \tau_s^{(r)} = \begin{cases} \tau_s \circ \sigma^{\tau_s^{(r-1)}} & \text{if } \tau_s^{(r-1)} < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Exercise: $\forall r \in \mathbb{N}_0 \ \forall s \in S \ \ \tau_s^{(r)}$ is a stopping time.

Given $s \in S$ and $r \in \mathbb{N}$, let

$$\lambda_s^{(r)} = \begin{cases} \tau_s^{(r)} - \tau_s^{(r-1)} & \text{if } \tau_s^{(r-1)} < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma

Let $r \geq 2$, $m \in \mathbb{N}$, $E \subseteq S$ non-empty. Then

$$\mu(\lambda_s^{(r)} = m \mid \tau_s^{(r-1)} < \infty, \pi_0 \in E) = \mu(\tau_s = m \mid \pi_0 = s)$$

Proof: Let $\mathcal{P}_1 = \{\{\tau_s^{(r-1)} < \infty\}, \{\tau_s^{(r-1)} = \infty\}\}$,

$\mathcal{P}_2 = \{\{\pi_0 \in E\}, \{\pi_0 \notin E\}\}$.

We need to evaluate $\mathbb{E}(\mathbb{1}_{\{\lambda_s^{(r)} = m\}} \mid \sigma(\mathcal{P}_1 \vee \mathcal{P}_2))$ on $A = \{\tau_s^{(r-1)} < \infty\} \cap \{\pi_0 \in E\}$.

Note: $\{\tau_s^{(r-1)} < \infty\} \in \mathcal{F}_{\tau_s^{(r-1)}}$ since $\{\tau_s^{(r-1)} < \infty\} \cap \{\tau_s^{(r-1)} \leq n\} = \{\tau_s^{(r-1)} \leq n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$,

$\{\pi_0 \in E\} \in \mathcal{F}_{\tau_s^{(r-1)}}$ since $\{\pi_0 \in E\} \in \mathcal{F}_0 \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}_0$.

Let $M = \mu(A)$, then

$$\mu(\lambda_s^{(r)} = m \mid \tau_s^{(r-1)} < \infty, \pi_0 \in E) = \frac{1}{M}\int_A \mathbb{1}_{\{\lambda_s^{(r)} = m\}}\, d\mu$$

$$= \frac{1}{M}\int_A \mathbb{1}_{\{\tau_s = m\}} \circ \sigma^{\tau_s^{(r-1)}}\, d\mu$$

$A \in \mathcal{F}_{\tau_s^{(r-1)}}$

$$= \frac{1}{M}\int_A \mathbb{E}\left(\mathbb{1}_{\{\tau_s = m\}} \circ \sigma^{\tau_s^{(r-1)}} \mid \mathcal{F}_{\tau_s^{(r-1)}}\right) d\mu$$

$$= \frac{1}{M}\int_A \mathbb{E}\left(\mathbb{1}_{\{\tau_s = m\}} \mid \mathcal{B}_0\right) \circ \sigma^{\tau_s^{(r-1)}}\, d\mu$$

$$= \frac{1}{M}\int_A \sum_{t \in S} \frac{\mathbb{E}\left(\mathbb{1}_{\{\tau_s = m, \pi_0 = t\}}\right)}{\mu(\mathcal{C}(t))} \mathbb{1}_{\mathcal{C}(t)} \circ \sigma^{\tau_s^{(r-1)}}\, d\mu$$

$\pi_0 \circ \sigma^{\tau_s^{(r-1)}} = s$ on $\{\tau_s^{(r-1)} < \infty\}$

$$= \frac{\mu(\tau_s = m, \pi_0 = s)}{\mu(\pi_0 = s)} \cdot \frac{1}{M}\int_A \mathbb{1}_{\mathcal{C}(s)} \circ \sigma^{\tau_s^{(r-1)}}\, d\mu = \mu(\tau_s = m \mid \pi_0 = s) \qquad \blacksquare$$

In what follows, given $s \in S$, let $V_s = \sum_{n=0}^{\infty} \mathbb{1}_{\{\pi_n = s\}}$ the number of visits to state s. Note that

$$\mathbb{E}(V_s \mid \pi_0 = s) = \sum_{n \geq 0} \mathbb{E}(\mathbb{1}_{\{\pi_n = s\}} \mid \pi_0 = s) = \sum_{n \geq 0} \mu(\pi_n = s \mid \pi_0 = s) = \sum_{n \geq 0} p_{ss}^{(n)}.$$

We also let $f_s = \mu(\tau_s < \infty \mid \pi_0 = s)$.

Lemma

Let $r \in \mathbb{N}_0$, then $\mu(V_s > r \mid \pi_0 = s) = f_s^r$

Proof: One verifies that $\{V_s > r, \pi_0 = s\} = \{\tau_s^{(r)} < \infty, \pi_0 = s\}$.

If $r = 0$, then $\mu(V_s > 0 \mid \pi_0 = s) = 1 = f_s^0$. So suppose $\mu(V_s > r \mid \pi_0 = s) = f_s^r$, then

$$\mu(V_s > r+1 \mid \pi_0 = s) = \mu(\tau_s^{(r+1)} < \infty \mid \pi_0 = s)$$

$$= \mu(\lambda_s^{(r+1)} < \infty, \tau_s^{(r)} < \infty \mid \pi_0 = s)$$

$$= \mu(\lambda_s^{(r+1)} < \infty \mid \tau_s^{(r)} < \infty, \pi_0 = s)\, \mu(\tau_s^{(r)} < \infty \mid \pi_0 = s)$$

$$= \mu(\tau_s < \infty \mid \pi_0 = s)\, \mu(V_s > r \mid \pi_0 = s) = f_s^r \qquad \blacksquare$$

Proof of the theorem (recurrence / transience dichotomy)

Note: Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ non-negative, then

$$(*) \quad \int_0^\infty \mathbb{P}(\{X > t\})\, dt = \int_0^\infty \int_\Omega \mathbb{1}_{\{X > t\}}\, d\mathbb{P}\, dt \overset{\text{Tonelli}}{=} \int_\Omega \int_0^\infty \mathbb{1}_{(0, X(\omega))}(t)\, dt\, d\mathbb{P}(\omega) = \int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \mathbb{E}(X).$$

Suppose first that $f_s = \mu(\tau_s < \infty \mid \pi_0 = s) = 1$, then

$$\mu(V_s = \infty \mid \pi_0 = s) = \lim_{r \to \infty} \mu(V_s > r \mid \pi_0 = s) = \lim_{r \to \infty} f_s^r = 1.$$

Hence, s is recurrent. Moreover

$$\sum_{n \geq 0} p_{ss}^{(n)} = \mathbb{E}(V_s \mid \pi_0 = s) \overset{(*)}{=} \sum_{r=0}^{\infty} \mu(V_s > r \mid \pi_0 = s) = \infty.$$

expectation with respect to $\mu(\cdot \mid \pi_0 = s)$

Now suppose that $f_s < 1$, then $\mu(V_s = \infty \mid \pi_0 = s) = \lim_{r \to \infty} f_s^r = 0$ and

$$\sum_{n \geq 0} p_{ss}^{(n)} = \sum_{r \geq 0} f_s^r = \frac{1}{1 - f_s} < \infty \qquad \blacksquare$$

Definition

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be a (p, P)-Markov chain. Let $s, t \in S$. We say that <u>s leads to t</u>, denoted $s \longrightarrow t$, if $\exists n \in \mathbb{N}_0$ such that

$$\mu(\pi_n = t \mid \pi_0 = s) > 0.$$

We say that s, t <u>communicate</u> if $s \longrightarrow t$ and $t \longrightarrow s$, and denote this by $s \longleftrightarrow t$.

End of lecture 18

Lemma

Let $(S^{\mathbb{N}}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be a (p, P)-Markov chain. Suppose $p_s > 0$. The following are equivalent.

(i) $s \longrightarrow t$.

(ii) $\exists n \in \mathbb{N}_0, s_0, \dots, s_n \in S$ such that $s_0 = s, s_n = t$, and $p_{s_0 s_1} \cdots p_{s_{n-1} s_n} > 0$.

(iii) $\mu\left(\bigcup_{n \geq 0} \{\pi_n = t\} \mid \pi_0 = s\right) > 0$.

Proof: (iii) $\Rightarrow$ (i) by subadditivity of $\mu(\cdot \mid \pi_0 = s)$

(i) $\Rightarrow$ (iii) by monotonicity of $\mu(\cdot \mid \pi_0 = s)$

(i) $\Leftrightarrow$ (ii) $\mu(\pi_n = t \mid \pi_0 = s) = \sum_{(s_1, \dots, s_{n-1}) \in S^{n-1}} \mu(\pi_1 = s_1, \dots, \pi_n = \overset{t}{\underset{\|}{s_n}} \mid \pi_0 = s) = \sum_{(s_1, \dots, s_{n-1}) \in S^{n-1}} \prod_{k=0}^{n-1} p_{s_k s_{k+1}}$ $\qquad \blacksquare$