Let $M=\mu(A)$, then

$$\mu(\lambda_s^{(r)}=m \mid \tau_s^{(r-1)}<\infty, \pi_0\in E) = \frac{1}{M}\int_A \mathbb{1}_{\{\lambda_s^{(r)}=m\}}\,d\mu$$

$$= \frac{1}{M}\int_A \mathbb{1}_{\{\tau_s=m\}}\circ\sigma^{\tau_s^{(r-1)}}\,d\mu$$

$$= \frac{1}{M}\int_A \mathbb{E}\left(\mathbb{1}_{\{\tau_s=m\}}\circ\sigma^{\tau_s^{(r-1)}} \,\middle|\, \mathcal{F}_{\tau_s^{(r-1)}}\right)d\mu$$

$$= \frac{1}{M}\int_A \mathbb{E}\left(\mathbb{1}_{\{\tau_s=m\}}\mid\mathcal{B}_0\right)\circ\sigma^{\tau_s^{(r-1)}}\,d\mu$$

$$= \frac{1}{M}\int_A \sum_{t\in S}\frac{\mathbb{E}(\mathbb{1}_{\{\tau_s=m,\pi_0=t\}})}{\mu(\mathcal{C}(t))}\mathbb{1}_{\mathcal{C}(t)}\circ\sigma^{\tau_s^{(r-1)}}\,d\mu$$

$$= \frac{\mu(\tau_s=m,\pi_0=s)}{\mu(\pi_0=s)}\cdot\frac{1}{M}\int_A \mathbb{1}_{\mathcal{C}(s)}\circ\sigma^{\tau_s^{(r-1)}}\,d\mu = \mu(\tau_s=m\mid\pi_0=s)$$ ∎

In what follows, given $s\in S$, let $V_s=\sum_{n=0}^{\infty}\mathbb{1}_{\{\pi_n=s\}}$ the number of visits to state s. Note that

$$\mathbb{E}(V_s\mid\pi_0=s)=\sum_{n\geq 0}\mathbb{E}(\mathbb{1}_{\{\pi_n=s\}}\mid\pi_0=s)=\sum_{n\geq 0}\mu(\pi_n=s\mid\pi_0=s)=\sum_{n\geq 0}p_{ss}^{(n)}.$$

We also let $f_s=\mu(\tau_s<\infty\mid\pi_0=s)$.

Lemma

Let $r\in\mathbb{N}_0$, then $\mu(V_s>r\mid\pi_0=s)=f_s^r$

Proof: One verifies that $\{V_s>r,\pi_0=s\}=\{\tau_s^{(r)}<\infty,\pi_0=s\}$.

If $r=0$, then $\mu(V_s>0\mid\pi_0=s)=1=f_s^0$. So suppose $\mu(V_s>r\mid\pi_0=s)=f_s^r$, then

$$\mu(V_s>r+1\mid\pi_0=s)=\mu(\tau_s^{(r+1)}<\infty\mid\pi_0=s)$$
$$=\mu(\lambda_s^{(r+1)}<\infty,\tau_s^{(r)}<\infty\mid\pi_0=s)$$
$$=\mu(\lambda_s^{(r+1)}<\infty\mid\tau_s^{(r)}<\infty,\pi_0=s)\,\mu(\tau_s^{(r)}<\infty\mid\pi_0=s)$$
$$=\mu(\tau_s<\infty\mid\pi_0=s)\,\mu(V_s>r\mid\pi_0=s)=f_s^r$$ ∎

Proof of the theorem (recurrence/transience dichotomy)

Note: Let $(\Omega,\mathcal{F},\mathbb{P})$ a probability space and $X\in\mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$ non-negative, then

$$(*)\quad \int_0^\infty \mathbb{P}(\{X>t\})\,dt = \int_0^\infty\int_\Omega \mathbb{1}_{\{X>t\}}\,d\mathbb{P}\,dt \overset{\text{Tonelli}}{=} \int_\Omega\int_0^\infty \mathbb{1}_{(0,X(\omega))}(t)\,dt\,d\mathbb{P}(\omega)=\int_\Omega X(\omega)\,d\mathbb{P}(\omega)=\mathbb{E}(X).$$

Suppose first that $f_s=\mu(\tau_s<\infty\mid\pi_0=s)=1$, then

$$\mu(V_s=\infty\mid\pi_0=s)=\lim_{r\to\infty}\mu(V_s>r\mid\pi_0=s)=\lim_{r\to\infty}f_s^r=1.$$

Hence, s is recurrent. Moreover

$$\sum_{n\geq 0}p_{ss}^{(n)}=\mathbb{E}(V_s\mid\pi_0=s)\overset{(*)}{=}\sum_{r=0}^{\infty}\mu(V_s>r\mid\pi_0=s)=\infty.$$

expectation with respect to $\mu(\cdot\mid\pi_0=s)$

$=f_s^r=1$

Now suppose that $f_s<1$, then $\mu(V_s=\infty\mid\pi_0=s)=\lim_{r\to\infty}f_s^r=0$ and

$$\sum_{n\geq 0}p_{ss}^{(n)}=\sum_{r\geq 0}f_s^r=\frac{1}{1-f_s}<\infty$$ ∎

Definition

Let $(S^{\mathbb{N}_0},\mathcal{B}(S^{\mathbb{N}_0}),\mu)$ be a (p,P)-Markov chain. Let $s,t\in S$. We say that <u>s leads to t</u>, denoted $s\longrightarrow t$, if $\exists n\in\mathbb{N}_0$ such that

$$\mu(\pi_n=t\mid\pi_0=s)>0$$

We say that s,t <u>communicate</u> if $s\longrightarrow t$ and $t\longrightarrow s$, and denote this by $s\longleftrightarrow t$.

End of lecture 18

Lemma

Let $(S^{\mathbb{N}},\mathcal{B}(S^{\mathbb{N}_0}),\mu)$ be a fully supported stationary (p,P)-Markov chain. The following are equivalent.

(i) $s\longrightarrow t$.

(ii) $\exists n\in\mathbb{N}_0, s_0,\dots,s_n\in S$ such that $s_0=s, s_n=t$, and $p_{s_0s_1}\cdots p_{s_{n-1}s_n}>0$.

(iii) $\mu\left(\bigcup_{n\geq 0}\{\pi_n=t\}\mid\pi_0=s\right)>0$.

Proof: (iii)$\Rightarrow$(i) by subadditivity of $\mu(\cdot\mid\pi_0=s)$

(i)$\Rightarrow$(iii) by monotonicity of $\mu(\cdot\mid\pi_0=s)$

(i)$\Leftrightarrow$(ii) $\mu(\pi_n=t\mid\pi_0=s)=\sum_{(s_1,\dots,s_n)\in S^{n-1}}\mu(\pi_1=s_1,\dots,\pi_n=\overset{t}{\overset{\shortparallel}{s_n}}\mid\pi_0=s)=\sum_{(s_1,\dots,s_{n-1})\in S^{n-1}}\prod_{k=0}^{n-1}p_{s_ks_{k+1}}$ ∎

Remark: Communication is an equivalence relation!

reflexivity: $\mu(\pi_0 = s \mid \pi_0 = s) = 1$.

transitivity: property (ii) in the lemma.

symmetry: by definition.

The equivalence classes with respect to this equivalence relation are called communicating classes.

Definition

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be a (p, P)-Markov chain. A (communicating) class $E \subseteq S$ is closed if $\forall s,t \in S \quad s \in E \wedge s \to t \Rightarrow t \in E$.

A state $s \in S$ is absorbing if $\{s\}$ is a closed communicating class.

The chain (or P) is irreducible if S is a single communicating class.

Theorem (recurrence and transience are class properties)

Let $s \in S$. The following are equivalent.

(i) s is recurrent.

(ii) $\forall t \in S \quad t \in [s] \Rightarrow t$ recurrent

Corollary

Let $s \in S$. Then s is transient if and only if every $t \in [s]$ is transient.

Proof: "$\Leftarrow$" clear

"$\Rightarrow$" Suppose s is transient and let $t \in [s]$. If t is recurrent, then so is s $\lightning$ ☒

Proof of theorem: (i) $\Rightarrow$ (ii): Suppose $t \in [s]$ is transient. Then $\sum_{n \geq 0} p_{tt}^{(n)} < \infty$. Since $t \in [s]$, there are $m_1, m_2 \in \mathbb{N}_0$ s.t. $p_{st}^{(m_1)} \cdot p_{ts}^{(m_2)} > 0$. For $r \in \mathbb{N}_0$, we have

$$p_{tt}^{(m_1+m_2+r)} \geq p_{ts}^{(m_2)} p_{ss}^{(r)} p_{st}^{(m_1)},$$

hence

$$\sum_{r \in \mathbb{N}_0} p_{ss}^{(r)} \leq \frac{1}{p_{st}^{(m_1)} \cdot p_{ts}^{(m_2)}} \sum_{n \geq m_1+m_2} p_{tt}^{(n)} < \infty \quad \lightning.$$

(ii) $\Rightarrow$ (i): Clear ☒

Theorem

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be a fully supported stationary (p, P)-Markov chain. Then every recurrent class is closed.

Proof: We show that non-closed classes are transient. Suppose $E \subseteq S$ is a non-closed class. Let $s \in E$, $t \in S - E$, $m \in \mathbb{N}_0$ s.t.

$$\mu(\pi_m = t \mid \pi_0 = s) > 0.$$

Claim:

$\mu(\{\pi_m = t\} \cap \limsup \{\pi_n = s\} \mid \pi_0 = s) = 0$

By assumption $t \in S - E$, i.e., $t \not\to s$. Hence

$$\forall n \in \mathbb{N}_0 \quad \mu(\pi_{m+n} = s \mid \pi_m = t) = \mu(\pi_n = s \mid \pi_0 = t) = 0.$$

$$\therefore \mu(\pi_{m+n} = s, \pi_m = t) = 0.$$

Now

$$\{\pi_m = t\} \cap \limsup \{\pi_n = s\} = \{\pi_m = t\} \cap \left(\bigcap_{n \geq m} \bigcup_{\ell \geq n} \{\pi_n = s\}\right)$$

$$= \bigcap_{n \geq m} \bigcup_{\ell \geq m+n} (\{\pi_m = t\} \cap \{\pi_{n+m} = s\})$$

is a μ-null set. It follows that

$$\mu(\limsup \{\pi_n = s\} \mid \pi_0 = s) = \sum_{t' \in S} \mu(\{\pi_m = t'\} \cap \limsup \{\pi_n = s\} \mid \pi_0 = s)$$

$$\leq \sum_{t' \in S - \{t\}} \mu(\pi_m = t' \mid \pi_0 = s) < 1.$$

☒

Theorem

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be a fully supported stationary (p, P)-Markov chain. Then every (finite) closed class is recurrent.

Proof: Let $E \subseteq S$ a closed class. Since E is finite and closed, there exists $s \in E$ such that

$$\mu(\underbrace{\pi_n = s \text{ for infinitely many } n \in \mathbb{N}}_{= \limsup \{\pi_n = s\}} \mid \pi_0 \in E) > 0.$$

Claim:

s is recurrent.

$$\mu(\limsup \{\pi_n = s\} \mid \pi_0 \in E) = \mu(\{\tau_s < \infty\} \cap \limsup \{\pi_n = s\} \mid \pi_0 \in E)$$

$$= \sum_{m \in \mathbb{N}} \mu(\{\tau_s = m\} \cap \limsup \{\pi_{m+n} = s\} \mid \pi_0 \in E)$$

$$\leq \sum_{m \in \mathbb{N}} \mu(\{\pi_m = s\} \cap \limsup \{\pi_{m+n} = s\} \mid \pi_0 \in E)$$

$$= \sum_{m\in\mathbb{N}} \mu\left(\limsup \{\pi_{m+n}=s\} \mid \pi_m = s, \pi_0 \in E\right) \mu(\pi_m = s \mid \pi_0 \in E)$$

Markov property

$$\overset{\downarrow}{=} \sum_{m\in\mathbb{N}} \mu\left(\limsup \{\pi_n = s\} \mid \pi_0 = s\right) \mu(\pi_m = s \mid \pi_0 \in E)$$

$$= \mu\left(\limsup \{\pi_n = s\} \mid \pi_0 = s\right) \sum_{m\in\mathbb{N}} \mu(\pi_m = s \mid \pi_0 \in E)$$

$$\Rightarrow \mu\left(\limsup \{\pi_n = s\} \mid \pi_0 = s\right) > 0.$$

∎

Theorem

Suppose $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ is an irreducible stationary (p, P)-Markov chain. Let $s \in S$. Then $\mu(\tau_s < \infty) = 1$.

Remark: Irreducibility $\Rightarrow p$ has full support by the exercise on Perron-Frobenius.

Proof: Note that $\mu(\tau_s < \infty) = \sum_{t\in S} \mu(\tau_s < \infty \mid \pi_0 = t)\mu(\pi_0 = t)$ implies that we need to show $\mu(\tau_s < \infty \mid \pi_0 = t) = 1$.

Using irreducibility, let $m \in \mathbb{N}_0$ such that $\mu(\pi_m = t \mid \pi_0 = s) > 0$. Since, by the previous theorem every state is recurrent, we have

$$1 = \mu\left(\limsup \{\pi_n = s\} \mid \pi_0 = s\right) = \mu\left(\limsup \{\pi_{m+n} = s\} \mid \pi_0 = s\right)$$

$$\le \mu\left(\bigcup_{n\in\mathbb{N}} \{\pi_{m+n} = s\} \mid \pi_0 = s\right) = \sum_{r\in S} \mu\left(\bigcup_{n\in\mathbb{N}} \{\pi_{m+n} = s\} \mid \pi_m = r, \pi_0 = s\right) \mu(\pi_m = r \mid \pi_0 = s)$$

Markov property

$$\overset{\downarrow}{=} \sum_{r\in S} \mu\left(\bigcup_{n\in\mathbb{N}} \{\pi_n = s\} \mid \pi_0 = r\right) \mu(\pi_m = r \mid \pi_0 = s)$$

$$= \sum_{r\in S} \mu(\tau_s < \infty \mid \pi_0 = r)\, \mu(\pi_m = r \mid \pi_0 = s).$$

Note: $w = \left(\mu(\pi_m = r \mid \pi_0 = s)\right)_{r\in S}$ is a probability vector. Thus, if $v \in [0,1]^S$ satisfies

$$\sum_{r\in S} v_r w_r \ge 1,$$

then $v_r = 1$ for all $r \in S$ for which $w_r \neq 0$ $\therefore$ $\mu(\tau_s < \infty \mid \pi_0 = t) = 1$. ∎

Theorem

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ be an irreducible stationary (p, P)-Markov chain. Then for $s \in S$

$$\mathbb{E}(\tau_s \mid \mathcal{F}_0)\big|_{\mathcal{C}(s)} = \frac{1}{p(s)}$$

Proof: $\mathbb{E}(\tau_s \mid \mathcal{F}_0)\big|_{\mathcal{C}(s)} = \frac{1}{p(s)} \mathbb{E}(\tau_s \cdot \mathbb{1}_{\mathcal{C}(s)}) = \frac{1}{p(s)} \int_{\mathcal{C}(s)} \tau_s \, d\mu = \frac{1}{p(s)}$

by Kac's lemma.

Theorem

Let $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \nu)$ be an irreducible (q, P)-Markov chain. Let $p \in [0,1]^S$ be the stationary probability measure on S, i.e., $pP = p$. Let μ be the shift-invariant measure induced by (p, P), i.e., $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu)$ is the stationary (p, P)-Markov chain. Then

$$\forall f \in L^1(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu) \quad \frac{1}{N}\sum_{n=0}^{N-1} f \circ \sigma^n \longrightarrow \int f \, d\mu \quad \nu\text{-a.s. and in } L^1(\nu)$$

Proof: By Birkhoff's pointwise ergodic theorem, we have (we will later show that $(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu, \sigma)$ is ergodic)

$$\frac{1}{N}\sum_{n=0}^{N-1} f \circ \sigma^n \longrightarrow \int f \, d\mu \quad \mu\text{-a.s.} \qquad (*)$$

Hence it suffices to show that $\nu \ll \mu$. But

$$\nu = \sum_{s\in S} q(s)\, \nu(\cdot \mid \pi_0 = s) = \sum_{s\in S} q(s)\, \mu(\cdot \mid \pi_0 = s), \quad \text{(and similarly for } \mu\text{)}$$

hence, if $X \subseteq S^{\mathbb{N}_0}$ has full measure such that $(*)$ is true on X, then

$$1 = \mu(X) = \sum_{s\in S} p(s)\, \mu(X \mid \pi_0 = s),$$

thus, using $p(s) > 0$ for all $s \in S$, $\mu(X \mid \pi_0 = s) = 1$ for all $s \in S$. Hence

$$\nu(S^{\mathbb{N}_0} \setminus X) = \sum_{s\in S} q(s)\, \mu(S^{\mathbb{N}_0} \setminus X \mid \pi_0 = s) = 0.$$

For L^1-convergence, note that $\int |f| \, d\mu = \sum_{s\in S} p(s) \int |f| \, d\mu(\cdot \mid \pi_0 = s)$ (details as exercise).