(iv)⇒(ii): Let $s, t \in S^{\ell+1}$. It suffices to show that

$$\mu(\mathcal{C}(s) \cap T^{-n}\mathcal{C}(t)) \longrightarrow \mu(\mathcal{C}(s))\mu(\mathcal{C}(t))$$

as $n \to \infty$, since this implies the characterization of strong mixing for simple functions, which are dense in $L^2(\mu)$.

But

$$\mu(\mathcal{C}(s) \cap T^{-n}\mathcal{C}(t)) = p(s_0) \cdot \prod_{k=0}^{\ell-1} P_{s_k s_{k+1}} \cdot (P^n)_{s_\ell t_0} \cdot \prod_{k=0}^{\ell-1} P_{t_k t_{k+1}}$$

$$\xrightarrow{n\to\infty} p(s_0) \prod_{k=0}^{\ell-1} P_{s_k s_{k+1}} \cdot p(t_0) \cdot \prod_{k=0}^{\ell-1} P_{t_k t_{k+1}} = \mu(\mathcal{C}(s))\mu(\mathcal{C}(t)).$$

$\blacksquare$

End of lecture 22

Invariant Measures for Continuous Transformations

In what follows, we will focus on the following setup. (X,d) will be a compact metric space and $\mathcal{B} = \mathcal{B}(X)$ is the Borel σ-algebra on X. We denote by $\mathcal{M}_1(X)$ the set of Borel probability measures on X. We recall that any $\mu \in \mathcal{M}_1(X)$ is regular, i.e.,

$$\forall B \in \mathcal{B}\ \forall \varepsilon > 0\ \exists C_\varepsilon \subseteq B \text{ compact } \exists U_\varepsilon \supseteq B \text{ open } \quad \mu(U_\varepsilon - C_\varepsilon) < \varepsilon.$$

In particular,

$$\forall B \in \mathcal{B} \quad \mu(B) = \inf_{\substack{B \subseteq U \\ U \text{ open}}} \mu(U) = \sup_{\substack{C \subseteq B \\ C \text{ compact}}} \mu(C).$$

We can map $\mathcal{M}_1(X)$ into $C(X)^*$ by defining

$$\forall \mu \in \mathcal{M}_1(X)\ \forall f \in C(X) \quad \langle f, \mu \rangle := \int f \, d\mu.$$

Then $\|\mu\|_{op} = 1$, since $\langle 1\!\!1_X, \mu \rangle = 1$ and $|\langle f, \mu \rangle| \leq \|f\|_\infty$.

This mapping is injective:

Theorem (without proof)

Let $\mu, \sigma \in \mathcal{M}_1(X)$. Then

$$\forall f \in C(X) \quad \langle f, \mu \rangle = \langle f, \sigma \rangle \implies \mu = \sigma.$$

cf. Walters, §6.1.

In fact, this mapping is onto in some sense, i.e., $C(X)^*$ identifies with the signed measures on $\mathcal{B}$. The main step in this direction and for us the crucial result is the following.

Theorem (Riesz–Kakutani–Markov, without proof)

Let $\ell : C(X) \to \mathbb{C}$ a bounded linear functional. Suppose that ℓ is positive, i.e., $\forall f \in C(X)\ f \geq 0 \Rightarrow \ell(f) \geq 0$, and normalized, i.e., $\ell(1\!\!1_X) = 1$. Then there exists $\mu \in \mathcal{M}_1(X)$ s.t.

$$\forall f \in C(X) \quad \ell(f) = \langle f, \mu \rangle.$$

Definition

The weak-* topology on $C(X)^*$ is the smallest topology s.t. for all $f \in C(X)$ the map

$$\begin{aligned} ev_f : C(X)^* &\longrightarrow \mathbb{C} \\ \ell &\longmapsto \ell(f) \end{aligned}$$

is continuous. The weak-* topology on $\mathcal{M}_1(X)$ is the restriction to $\mathcal{M}_1(X)$ of the weak-* topology on $C(X)^*$.

Remark: The weak-* topology on $\mathcal{M}_1(X)$ is generated by the open neighbourhoods of the form

$$N_{f_1,\dots,f_s;\varepsilon}(\mu) = \bigcap_{i=1}^{s} \left\{ \sigma \in \mathcal{M}_1(X) : \left| \int f_i \, d\mu - \int f_i \, d\sigma \right| < \varepsilon \right\},$$

where $s \in \mathbb{N}$, $\varepsilon > 0$, $f_1, \dots, f_s \in C(X)$, $\mu \in \mathcal{M}_1(X)$ vary.

Theorem (exercise)

Let $(f_n)_{n \in \mathbb{N}}$ a dense sequence in $C(X) - \{0\}$ and define

$$\begin{aligned} D : \mathcal{M}_1(X) \times \mathcal{M}_1(X) &\longrightarrow [0, \infty) \\ (\sigma, \mu) &\longmapsto \sum_{n=1}^{\infty} \frac{|\langle f_n, \sigma \rangle - \langle f_n, \mu \rangle|}{2^n \|f_n\|_\infty}. \end{aligned}$$

Then D is a metric inducing the weak-* topology on $\mathcal{M}_1(X)$.

Theorem (Special case of Banach–Alaoglu)

$\overline{B}_1^{C(X)^*}(0) \subseteq C(X)^*$ is compact in the weak-* topology.

Lemma

Let $T: X \to X$ continuous, $U_T^*: \mathcal{M}_1(X) \to \mathcal{M}_1(X)$. Then U_T^* is continuous
$$\mu \mapsto T_*\mu$$
in the weak-* topology.

Proof: Suppose $\mu_n \to \mu$ in the weak-* topology. Let $f \in C(X)$ arbitrary. Then $f \circ T \in C(X)$, hence
$$\langle f, U_T^*\mu_n \rangle = \langle U_T f, \mu_n \rangle \xrightarrow{n \to \infty} \langle U_T f, \mu \rangle = \langle f, U_T^*\mu \rangle,$$
thus $U_T^*\mu = \lim_{n\to\infty} U_T^*\mu_n$. ☐

Lemma

$\mathcal{M}_1(X) \subseteq C(X)^*$ is compact in the weak-* topology.

Proof: We have seen that $\mathcal{M}_1(X) \subseteq \overline{B_1^{C(X)^*}(0)}$, thus it suffices to show that $\mathcal{M}_1(X)$ is closed. By the Riesz representation theorem
$$\mathcal{M}_1(X) = ev_{1\!\!1_X}^{-1}(\{1\}) \cap \bigcap_{f \geq 0} ev_f^{-1}([0,\infty)),$$
which is closed by definition of the weak-* topology.

Proposition:

Let $T: X \to X$ continuous. Then
$$\mathcal{M}_1^T(X) = \{\mu \in \mathcal{M}_1(X) : T_*\mu = \mu\}$$
is a compact, convex subset of $\mathcal{M}_1(X)$.

Proof: · Convexity is clear

· Compactness: $\mathcal{M}_1^T(X) = \bigcap_{f \in C(X)} \left(ev_{f - f\circ T}(\{0\}) \cap \mathcal{M}_1(X) \right)$ is a closed subset of $\mathcal{M}_1(X)$, hence compact.

Theorem (Kryloff-Bogolioubboff)

Let $\{\sigma_N : N \in \mathbb{N}\} \subseteq \mathcal{M}_1(X)$ and $\mu_N := \frac{1}{N}\sum_{n=0}^{N-1} T_*^n \sigma_N$, then any weak-* limit of the sequence $(\mu_N)_{N\in\mathbb{N}}$ is T-invariant. In particular, $\mathcal{M}_1^T(X) \neq \emptyset$.

Proof: Let $\mu \in C(X)^*$ s.t. $\mu_{N_k} \to \mu$ in the weak-* topology as $k \to \infty$ for some $(N_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ with $N_k \nearrow \infty$. Since $\mathcal{M}_1(X)$ is compact, we know that $\mu \in \mathcal{M}_1(X)$. Moreover, for any $f \in C(X)$
$$|\mu(f) - T_*\mu(f)| = \left|\lim_{k\to\infty} \mu_{N_k}(f) - \mu_{N_k}(f\circ T)\right|$$
$$= \lim_{k\to\infty} \left| \frac{1}{N_k}\left(\sigma_{N_k}(f) - \sigma_{N_k}(f\circ T^{N_k})\right)\right|$$
$$\leq \lim_{k\to\infty} \frac{2\|f\|_\infty}{N_k} = 0.$$

Since $\mathcal{M}_1(X)$ is compact, metric, any sequence has a converging subsequence and, since $\mathcal{M}_1(X) \supseteq \{\delta_x : x \in X\}$ is non-empty, $\mathcal{M}_1^T(X) \neq \emptyset$. ☐

Exercise: · $\mu \in \mathcal{M}_1^T(X)$ is extreme $\iff$ μ is ergodic

· Suppose $\mu, \sigma \in \mathcal{M}_1^T(X)$ are ergodic. Then
$$\mu \neq \sigma \iff \mu \perp \sigma.$$

Exercise:

Let $T: X \to X$ continuous, $\mu \in \mathcal{M}_1^T(X)$.

(i) μ is ergodic if and only if for all $\sigma \in \mathcal{M}_1(X)$
$$\sigma \ll \mu \implies \frac{1}{N}\sum_{n=0}^{N-1} T^n{}_*\sigma \xrightarrow{N\to\infty} \mu.$$

(ii) μ is (strong) mixing if and only if for all $\sigma \in \mathcal{M}_1(X)$
$$\sigma \ll \mu \implies T^n{}_*\sigma \xrightarrow{n\to\infty} \mu.$$

(iii) μ is weak-mixing if and only if $\exists J \subseteq \mathbb{N}$ density zero such that for all $\sigma \in \mathcal{M}_1(X)$
$$\sigma \ll \mu \implies T^n_*\sigma \longrightarrow \mu \text{ as } n \to \infty \text{ in } \mathbb{N} - J.$$

Exercise

Let $T: X \to X$ continuous, $\mu \in \mathcal{M}_1^T(X)$. Then
$$\mu \text{ is ergodic} \iff \frac{1}{N}\sum_{n=0}^{N-1} T^n{}_*\delta_x \xrightarrow{N\to\infty} \mu \quad \mu\text{-a.s.}$$

Definition

Let $T: X \to X$ continuous. T is <u>uniquely ergodic</u> if $|\mathcal{M}_1^T(X)| = 1$.

Theorem

Let $T: X \to X$ a homeomorphism and suppose T is uniquely ergodic with unique invariant measure μ. Then T is minimal if and only if for all non-empty open subsets $U \subseteq X$ we have $\mu(U) > 0$, i.e., μ has full support.

Proof: "$\Rightarrow$" Suppose T is minimal. If $U \subseteq X$ open non-empty, then

$$X = \bigcup_{n\in\mathbb{Z}} T^n U,$$

hence

$$1 = \mu(X) \le \sum_{n\in\mathbb{Z}} \mu(T^n U) = \sum_{n\in\mathbb{Z}} \mu(U)$$

and, therefore, $\mu(U) > 0$.

"$\Leftarrow$" Suppose T isn't minimal and let $E \subseteq X$ closed s.t. $TE = E$ and $E \notin \{\emptyset, X\}$. Then $T|_E : E \to E$ is a homeomorphism of a compact metric space and, hence, admits an invariant probability measure μ_E. Define $\tilde{\mu} \in \mathcal{M}_1^T(X)$ by $\forall B \in \mathcal{B}$ $\tilde{\mu}(B) = \mu_E(B \cap E)$. Since T is uniquely ergodic, we have $\tilde{\mu} = \mu$ and, hence, $\mu(X - E) = \mu_E((X-E) \cap E) = 0$, i.e., there exists $U\,(= X - E) \subseteq X$ non-empty open such that $\mu(U) = 0$. ☒

Theorem (Fürstenberg)

Let $T: \mathbb{T} \to \mathbb{T}$ be a homeomorphism without periodic points. Then T is uniquely ergodic. Moreover

(i) $\exists \varphi: \mathbb{T} \to \mathbb{T}$ continuous surjective and $\alpha \in \mathbb{R} - \mathbb{Q}$ s.t.

$$\begin{tikzcd} \mathbb{T} \arrow[r, "T"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{T} \arrow[d, "\varphi"] \\ \mathbb{T} \arrow[r, "R_\alpha"'] & \mathbb{T} \end{tikzcd}$$

and, for every $t \in \mathbb{T}$, $\varphi^{-1}(\{t\})$ is either a point or a sub-interval.

(ii) If T is minimal, then φ is a homeomorphism, i.e., every minimal homeomorphism is topologically conjugate to an irrational rotation.

Lemma

Let (X,d) compact metric, $T: X \to X$ continuous. Let $N \in \mathbb{N}$ and $x \in X$. Then $T^N x = x$ if and only if $\frac{1}{N}\sum_{n=0}^{N-1} T^n_* \delta_x \in \mathcal{M}_1^T(X)$.

Proof: Let $\mu_{N,x} = \frac{1}{N}\sum_{n=0}^{N-1} U_T^{*n}\delta_x$. Then $\mu_{N,x} \in \mathcal{M}_1(X)$ and

$$\mu_{N,x} \in \mathcal{M}_1^T(X) \iff \forall f \in C(X) \quad \mu_{N,x}(f) = \mu_{N,x}(f \circ T)$$

$$\iff \forall f \in C(X) \quad f(x) = f(T^N x).$$

Since $C(X)$ separates points, it follows that

$$\mu_{N,x} \in \mathcal{M}_1^T(X) \iff x = T^N x.$$

☒

Proof of the theorem: We start with (i)

Step 1: Let $\mu \in \mathcal{M}_1^T(\mathbb{T})$, then

$$\forall x \in \mathbb{T} \quad \mu(\{x\}) = 0.$$

Suppose $\mu(\{x\}) > 0$, then

$$1 \ge \mu(\mathcal{O}(x)) = \sum_{y\in\mathcal{O}(x)} \mu(\{y\}) = \mu(\{x\})\,|\mathcal{O}(x)| \Rightarrow |\mathcal{O}(x)| < \infty$$

$\therefore \exists N \in \mathbb{N}$ $T^N x = x$.

Step 2: Let $\mu \in \mathcal{M}_1^T(\mathbb{T})$, then the map $\underset{\in \mathbb{T}}{t} \longmapsto \mu([0,t] \bmod \mathbb{Z}) \bmod \mathbb{Z} \in \mathbb{T}$ is continuous.

Suppose $t \in \mathbb{T} - \{0\}$ and let $t_n \nearrow t$. Then

$$\mu([0,t] \bmod \mathbb{Z}) - \mu([0,t_n] \bmod \mathbb{Z}) = \mu((t_n, t] \bmod \mathbb{Z}) \xrightarrow{n\to\infty} \mu(\{t\}) = 0$$

since $E_n := (t_n, t]$ is a nested sequence and $\bigcap_{n\in\mathbb{N}} E_n = \{t\}$.

Similarly, one argues for $t_n \searrow t$.

If $t = 0 \in \mathbb{T}$, then we use $\mu([0,1]) = 1 \equiv 0 \bmod \mathbb{Z} = \mu(\{0\} \bmod \mathbb{Z}) \bmod \mathbb{Z}$ and separately verify that $t_n \nearrow 1 \Rightarrow \mu([0,t_n] \bmod \mathbb{Z}) \to 1$ and $t_n \searrow 0 \Rightarrow \mu([0,t_n] \bmod \mathbb{Z}) \to 0$.

Step 3: Use $\varphi(t) = \mu([0,t] \bmod \mathbb{Z}) \bmod \mathbb{Z}$.

Denote $\varphi : \mathbb{T} \longrightarrow \mathbb{T}$

$$t \longmapsto \mu([0,t] \bmod \mathbb{Z}) \bmod \mathbb{Z},$$

so that φ is continuous and surjective (since $t \longmapsto \mu([0,t] \bmod \mathbb{Z})$ is surjective onto $[0,1]$).

Note that for all $t_1, t_2, t_3 \in \mathbb{T}$ we have

$$\mu([t_1,t_2] \bmod \mathbb{Z}) + \mu([t_2,t_3] \bmod \mathbb{Z}) \equiv \mu([t_1,t_3] \bmod \mathbb{Z}) \bmod \mathbb{Z},$$

where for $a, b \in \mathbb{R}$, we let

$$\mu([a,b] \bmod \mathbb{Z}) = \begin{cases} \mu([\{a\},\{b\}] \bmod \mathbb{Z}) & \text{if } \{a\} \leq \{b\}, \\ \mu([\{b\},1] \bmod \mathbb{Z}) + \mu([0,\{a\}] \bmod \mathbb{Z}) & \text{otherwise}. \end{cases}$$

Hence

$$\varphi \circ T(t) = \mu([0,T(t)] \bmod \mathbb{Z}) \bmod \mathbb{Z}$$

$$= \mu([T(0),T(t)] \bmod \mathbb{Z}) + \mu([0,T(0)] \bmod \mathbb{Z}) \bmod \mathbb{Z}$$

$$= \varphi(t) + \mu([0,T(0)] \bmod \mathbb{Z}) \bmod \mathbb{Z}.$$

We need to show that $\underbrace{\mu([0,T(0)] \bmod \mathbb{Z})}_{=:\alpha}$ is irrational. Assume otherwise and suppose $q\alpha \in \mathbb{Z}$. Then $\varphi \circ T^q = \varphi$ and, hence,

$$\forall t \in \mathbb{T} \quad \mu([0,t] + \mathbb{Z}) = \mu([0,T^q t] + \mathbb{Z})$$

$$\Rightarrow \forall t \in \mathbb{T} \quad \mu([t,T^q t] + \mathbb{Z}) = 0.$$

Since $T^q : \mathbb{T} \to \mathbb{T}$ is continuous, $\mathbb{T}$ is compact, and $T^q t \neq t$ for all $t \in \mathbb{T}$, we have

$$\inf\{d(t,T^q t) : t \in \mathbb{T}\} = \min\{d(t,T^q t) : t \in \mathbb{T}\} > 0$$

and there is $m \in \mathbb{N}$ such that for all $t \in \mathbb{T}$

$$d(t, T^q t) > \frac{2}{m}.$$

Since T is orientation preserving (otherwise T has a fixed point), we have that

$$\mathbb{T} = \bigcup_{k=0}^{m-1} \left(\left[\tfrac{k}{m}, T^q\left(\tfrac{k}{m}\right)\right] \bmod \mathbb{Z}\right)$$

and, hence, $\mu(\mathbb{T}) = 0$, which is absurd.

Therefore

$$\varphi \circ T = R_\alpha \circ \varphi$$

with $\alpha \in \mathbb{R} - \mathbb{Q}$ as desired.

End of lecture 23

Next, we deduce unique ergodicity. To this end, note that

$$\forall \mu \in \mathcal{M}_1^T(\mathbb{T}) \quad \varphi_* \mu \in \mathcal{M}_1^{R_\alpha}(\mathbb{T}) = \{\mathrm{Leb}\},$$

since R_α is uniquely ergodic.

Claim:

Let $t \in \mathbb{T}$, then $\varphi^{-1}(\{\varphi(t)\}) = [t_-, t_+] + \mathbb{Z}$, where

$$t_- = \inf\{\tau \in [0,1] : \mu([\tau,t] + \mathbb{Z}) \equiv 0 \bmod \mathbb{Z}\},$$

$$t_+ = \sup\{\tau \in [0,1] : \mu([t,\tau] + \mathbb{Z}) \equiv 0 \bmod \mathbb{Z}\}.$$

Indeed, $\tau \in \varphi^{-1}(\{\varphi(t)\})$ iff $\mu([\tau,t] \bmod \mathbb{Z}) \equiv 0 \bmod \mathbb{Z}$. Hence the claim follows from monotonicity of μ and non-atomicity. This implies the claim about $\varphi^{-1}(\{t\})$.

⌜Suppose $\varphi(t) \in (0,1)$, then

$$\varphi(\tau) = \varphi(t) \Leftrightarrow \mu([0,t] + \mathbb{Z}) = \mu([0,\tau] + \mathbb{Z})$$

$$\Leftrightarrow \mu([\tau,t] + \mathbb{Z}) \in \{0,1\}.$$

Suppose $\varphi(t) \in \{0,1\}$, then

$$\varphi(\tau) = \varphi(t) \Leftrightarrow \mu([0,\tau] + \mathbb{Z}) \in \{0,1\}$$

$$\Leftrightarrow \mu([\tau,1] + \mathbb{Z}) \in \{0,1\}$$ ⌟

Now, for unique ergodicity, let $0 < a < b < 1$, then $\varphi([a,b] + \mathbb{Z}) = [\varphi(a), \varphi(b)] \bmod \mathbb{Z}$ by "monotonicity" of $\varphi|_{(0,1)}$, hence $\varphi^{-1}\varphi([a,b] + \mathbb{Z}) = [a_-, b_+]$ and, thus,

$$\mu([a,b] + \mathbb{Z}) = \mu([a_-, b_+]) = \mathrm{Leb}(\varphi([a,b] + \mathbb{Z}))$$

is independent of μ. ⌜$\mu([a_-,b_+] \Delta [a,b]) = 0$ by definition of a_- and b_+⌟

It remains to prove (ii). So suppose T is minimal. We know that, letting $\mu \in \mathcal{M}_1^T(\mathbb{T})$, for all $\emptyset \neq U \subseteq \mathbb{T}$ open $\mu(U) > 0$. Since $\mathbb{T}$ is compact, φ a continuous surjection, we need to show that φ is injective. Let $a < b$ in $[0,1)$, then $\mu([a,b] + \mathbb{Z}) > 0$, hence

$$\mu([0,b] + \mathbb{Z}) > \mu([0,a] + \mathbb{Z})$$

and, hence, $\varphi(a \bmod \mathbb{Z}) = \varphi(b \bmod \mathbb{Z})$ implies $\mu([0,a] + \mathbb{Z}) = 0$ and $\mu([0,b] + \mathbb{Z}) = 1$. Using the same argument again, we have $\mu([b,1] + \mathbb{Z}) > 0$ (since $b < 1$ by assumption), hence $\mu([0,b] + \mathbb{Z}) \neq 1$ and, thus,

$$\varphi(a \bmod \mathbb{Z}) \neq \varphi(b \bmod \mathbb{Z}).$$

$\boxtimes$