Denote $\varphi : \mathbb{T} \longrightarrow \mathbb{T}$

$$t \longmapsto \mu([0,t] \bmod \mathbb{Z}) \bmod \mathbb{Z},$$

so that φ is continuous and surjective (since $t \longmapsto \mu([0,t] \bmod \mathbb{Z})$ is surjective onto $[0,1]$).

Note that for all $t_1, t_2, t_3 \in \mathbb{T}$ we have

$$\mu([t_1,t_2] \bmod \mathbb{Z}) + \mu([t_2,t_3] \bmod \mathbb{Z}) \equiv \mu([t_1,t_3] \bmod \mathbb{Z}) \bmod \mathbb{Z},$$

where for $a, b \in \mathbb{R}$, we let

$$\mu([a,b] \bmod \mathbb{Z}) = \begin{cases} \mu([\{a\},\{b\}] \bmod \mathbb{Z}) & \text{if } \{a\} \leq \{b\}, \\ \mu([\{b\},1] \bmod \mathbb{Z}) + \mu([0,\{a\}] \bmod \mathbb{Z}) & \text{otherwise}. \end{cases}$$

Hence

$$\varphi \circ T(t) = \mu([0,T(t)] \bmod \mathbb{Z}) \bmod \mathbb{Z}$$

$$= \mu([T(0),T(t)] \bmod \mathbb{Z}) + \mu([0,T(0)] \bmod \mathbb{Z}) \bmod \mathbb{Z}$$

$$= \varphi(t) + \mu([0,T(0)] \bmod \mathbb{Z}) \bmod \mathbb{Z}.$$

We need to show that $\underbrace{\mu([0,T(0)] \bmod \mathbb{Z})}_{=:\alpha}$ is irrational. Assume otherwise and suppose $q\alpha \in \mathbb{Z}$. Then $\varphi \circ T^q = \varphi$ and, hence,

$$\forall t \in \mathbb{T} \quad \mu([0,t]+\mathbb{Z}) = \mu([0,T^q t]+\mathbb{Z})$$

$$\Rightarrow \forall t \in \mathbb{T} \quad \mu([t,T^q t]+\mathbb{Z}) = 0.$$

Since $T^q : \mathbb{T} \longrightarrow \mathbb{T}$ is continuous, $\mathbb{T}$ is compact, and $T^q t \neq t$ for all $t \in \mathbb{T}$, we have

$$\inf\{d(t,T^q t) : t \in \mathbb{T}\} = \min\{d(t,T^q t) : t \in \mathbb{T}\} > 0$$

and there is $m \in \mathbb{N}$ such that for all $t \in \mathbb{T}$

$$d(t, T^q t) > \frac{2}{m}.$$

Since T is orientation preserving (otherwise T has a fixed point), we have that

$$\mathbb{T} = \bigcup_{k=0}^{m-1} \left(\left[\tfrac{k}{m}, T^q\left(\tfrac{k}{m}\right)\right] \bmod \mathbb{Z}\right)$$

and, hence, $\mu(\mathbb{T}) = 0$, which is absurd.

Therefore

$$\varphi \circ T = R_\alpha \circ \varphi$$

with $\alpha \in \mathbb{R} - \mathbb{Q}$ as desired.

End of lecture 23

Next, we deduce unique ergodicity. To this end, note that

$$\forall \mu \in \mathcal{M}_1^T(\mathbb{T}) \qquad \varphi_* \mu \in \mathcal{M}_1^{R_\alpha}(\mathbb{T}) = \{\mathrm{Leb}\},$$

since R_α is uniquely ergodic. Let $\mu_1, \mu_2 \in \mathcal{M}_1^T(\mathbb{T})$ and suppose $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$.

<u>Claim:</u>

Let $t \in \mathbb{T}$, then $\varphi^{-1}(\{\varphi(t)\}) = [t_-, t_+] + \mathbb{Z}$, where

$$t_- = \inf\{\tau \in [0,1] : \mu([\tau,t]+\mathbb{Z}) \equiv 0 \bmod \mathbb{Z}\},$$

$$t_+ = \sup\{\tau \in [0,1] : \mu([t,\tau]+\mathbb{Z}) \equiv 0 \bmod \mathbb{Z}\}.$$

Indeed, $\tau \in \varphi^{-1}(\{\varphi(t)\})$ iff $\mu([\tau,t] \bmod \mathbb{Z}) \equiv 0 \bmod \mathbb{Z}$. Hence the claim follows from monotonicity of μ and non-atomicity. This implies the claim about $\varphi^{-1}(\{t\})$.

⌜Suppose $\varphi(t) \in (0,1)$, then

$\varphi(\tau) = \varphi(t) \iff \mu([0,t]+\mathbb{Z}) = \mu([0,\tau]+\mathbb{Z})$

$\iff \mu([\tau,t]+\mathbb{Z}) \in \{0,1\}$.

Suppose $\varphi(t) \in \{0,1\}$, then

$\varphi(\tau) = \varphi(t) \iff \mu([0,\tau]+\mathbb{Z}) \in \{0,1\}$

$\iff \mu([\tau,t]+\mathbb{Z}) \in \{0,1\}$⌟

Now, for unique ergodicity, let $0 < a < b < 1$, then $\varphi([a,b]+\mathbb{Z}) = [\varphi(a), \varphi(b)] \bmod \mathbb{Z}$ by "monotonicity" of $\varphi|_{(0,1)}$, hence $\varphi^{-1}\varphi([a,b]+\mathbb{Z}) = [a_-, b_+]$ and, thus, for $i = 1,2$

$$\mu_i([a,b]+\mathbb{Z}) \overset{\mu_i \ll \mu}{=} \mu_i([a_-,b_+]) = \mathrm{Leb}(\varphi([a,b]+\mathbb{Z}))$$

is independent of i. ⌜$\mu([a_-,b_+] \Delta [a,b]) = 0$ by definition of a_- and b_+⌟

It remains to prove (ii). So suppose T is minimal. We know that, letting $\mu \in \mathcal{M}_1^T(\mathbb{T})$, for all $\emptyset \neq U \subseteq \mathbb{T}$ open $\mu(U) > 0$. Since $\mathbb{T}$ is compact, φ a continuous surjection, we need to show that φ is injective. Let $a < b$ in $[0,1)$, then $\mu([a,b]+\mathbb{Z}) > 0$, hence

$$\mu([0,b]+\mathbb{Z}) > \mu([0,a]+\mathbb{Z})$$

and, hence, $\varphi(a \bmod \mathbb{Z}) = \varphi(b \bmod \mathbb{Z})$ implies $\mu([0,a]+\mathbb{Z}) = 0$ and $\mu([0,b]+\mathbb{Z}) = 1$. Using the same argument again, we have $\mu([b,1]+\mathbb{Z}) > 0$ (since $b < 1$ by assumption), hence $\mu([0,b]+\mathbb{Z}) \neq 1$ and, thus,

$$\varphi(a \bmod \mathbb{Z}) \neq \varphi(b \bmod \mathbb{Z}).$$

$\boxtimes$

In what follows, we want to give an abstract characterization of unique ergodicity. To this end we need some more functional analysis.

Definition

Let V be a locally convex topological vector space (e.g. $C(X)^*$) and $A \subseteq V$. The closed convex hull of A is the smallest closed convex subset of V containing A (or equivalently the closure of the set of all convex combinations of elements in A.

Theorem (Krein-Milman)

Let V be a locally convex topological vector space over $\mathbb{R}$ and let $K \subseteq V$ compact convex. Then K is the closed convex hull of the set of extreme points in K. In particular, if K is non-empty, then $\mathrm{ext}(K)$ is non-empty.

Theorem (Choquet)

Let V be a locally convex topological vector space over $\mathbb{R}$ and let $K \subseteq V$ compact convex and metrizable. Then $\mathrm{ext}(K) \subseteq K$ is Borel measurable and for every $v \in K$ there exists a Borel probability measure σ on K s.t. $\sigma(\mathrm{ext}(K)) = 1$ and

$$\forall \ell \in V^* \qquad \ell(v) = \int_K \ell(w)\, d\sigma(w).$$

Corollary (Ergodic decomposition)

Let X compact metric, $T: X \to X$ continuous, $\mu \in \mathcal{M}_1^T(X)$. There exists a (unique) Borel probability measure σ on $(\mathcal{M}_1^T(X), \mathcal{B}(\mathcal{M}_1^T(X)))$ such that $\mu(\mathcal{E}^T(X)) = 1$ and such that for all $f \in C(X)$

$$\int f\, d\mu = \int_{\mathcal{E}^T(X)} \left(\int_X f(x)\, d\lambda(x) \right) d\sigma(\lambda)$$

(i.e. $\mu = \int \lambda\, d\sigma(\lambda)$)

Theorem

Let (X,d) compact metric, $T: X \to X$ continuous. The following are equivalent.

(i) T is uniquely ergodic.

(ii) $|\mathcal{E}^T(X)| = 1$.

(iii) $\forall f \in C(X)\ \exists C_f \in \mathbb{C}\ \forall x \in X \quad A_N f(x) \longrightarrow C_f$.

(iv) $\forall f \in C(X)\ \exists C_f \in \mathbb{C}\ \forall x \in X \quad A_N f(x) \longrightarrow C_f$ uniformly.

(v) $\exists E \subseteq C(X)$ dense $\forall f \in E\ \exists C_f \in \mathbb{C}\ \forall x \in X \quad A_N f(x) \longrightarrow C_f$.

Proof: (i) $\Leftrightarrow$ (ii) by Krein-Milman and Choquet.

(i) $\Rightarrow$ (iii) If T is uniquely ergodic with unique invariant probability measure μ, if $x \in X$ is arbitrary, then every converging subsequence of (in weak-* topology)

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n x}$$

converges to μ. Hence for every $x \in X$

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n x} \xrightarrow[\text{(weak-*)}]{N \to \infty} \mu.$$

(iii) $\Rightarrow$ (i) Let $f \in C(X)$ arbitrary. Since f is bounded, Lebesgue's dominated convergence yields

$$\int f\, d\mu = \int \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)\, d\mu(x) = C_f.$$

Since μ is completely characterized by the values $\int f\, d\mu$, the claim follows.

(i) $\Rightarrow$ (iv) Exercise.

Hint:

$\neg$ uniform $\Leftrightarrow \exists \varepsilon > 0\ \exists f \in C(X)\ \exists (N_k)_{k \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}, N_k \nearrow \infty, \exists (x_k)_{k \in \mathbb{N}} \in X^{\mathbb{N}}$

$$\forall k \in \mathbb{N} \quad \left| \frac{1}{N_k} \sum_{n=0}^{N_k - 1} f(T^n x_k) - \int f\, d\mu \right| \geq \varepsilon.$$

(iv) $\Rightarrow$ (v) is clear.

(v) $\Rightarrow$ (i) Let $\mu, \sigma \in \mathcal{M}_1^T(X)$. Then by Lebesgue's dominated convergence

$$\forall f \in E \quad \int f\, d\mu = \int \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)\, d\mu(x) = C_f$$

$$= \int \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)\, d\sigma(x) = \int f\, d\sigma.$$

Since $E \subseteq C(X)$ is dense, $\mu = \sigma$.

$\blacksquare$

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