7. A few words about topological entropy and some words about EKL06

Definition

Let (X,d) compact metric, $\mathcal{T}\subseteq 2^X$ the topology induced by d, and $\mathcal{U}\subseteq\mathcal{T}$ a cover of X. Then
$$N(\mathcal{U}):=\min\{|\mathcal{V}|:\mathcal{V}\subseteq\mathcal{U}\text{ is a cover of }X\}$$

Definition

Let $\mathcal{U},\mathcal{V}\subseteq\mathcal{T}$ open covers, then
$$\mathcal{U}\vee\mathcal{V}:=\{U\cap V: U\in\mathcal{U}, V\in\mathcal{V}\}$$

Remark:

- Let $\mathcal{U},\mathcal{V}\subseteq\mathcal{T}$ covers of X, then $N(\mathcal{U}\vee\mathcal{V})\le N(\mathcal{U})N(\mathcal{V})$.
- Let $\mathcal{U}\subseteq\mathcal{T}$ a cover of X, $T:X\longrightarrow X$ continuous. Then
$$T^{-1}\mathcal{U}:=\{T^{-1}U: U\in\mathcal{U}\}\subseteq\mathcal{T}$$
is a cover of X and $N(T^{-1}\mathcal{U})\le N(\mathcal{U})$.
- Let $\mathcal{U}$ a cover of X, then for $m\le n$ $\mathcal{U}_m^n:=T^{-m}\mathcal{U}\vee\cdots\vee T^{-n}\mathcal{U}$.

Definition

Let (X,d) compact, metric, $T:X\longrightarrow X$ continuous. The <u>topological entropy</u> of T with respect to the open cover $\mathcal{U}\subseteq\mathcal{T}$ is
$$h_{top}(T,\mathcal{U})=\lim_{N\to\infty}\frac{1}{N}\log N(\mathcal{U}_0^{N-1}).$$

Remark: The limit exists by subadditivity of $a_N:=\log N(\mathcal{U}_0^{N-1})$.

- This is sometimes called the <u>cover entropy</u> of $\mathcal{U}$.

Definition

Let (X,d) compact metric, $T:X\longrightarrow X$ continuous. The <u>topological entropy</u> of T is
$$h_{top}(T)=\sup\{h_{top}(T,\mathcal{U}):\mathcal{U}\subseteq\mathcal{T}\text{ cover } X\}$$

Definition

Let (X,d) compact metric, $T:X\longrightarrow X$ continuous.
- $E\subseteq X$ is <u>(n,ε)-separated</u> if $\forall x,y\in E$ distinct $\max\{d(T^kx,T^ky):0\le k<n\}\ge\varepsilon$.
- $F\subseteq X$ is <u>(n,ε)-dense</u> if $\forall x\in X\,\exists y\in F$ s.t. $\max\{d(T^kx,T^ky):0\le k<n\}<\varepsilon$

We let $s_n(T,\varepsilon)=\max\{|E|: E\text{ is }(n,\varepsilon)\text{-separated}\}$, $s(T,\varepsilon)=\overline{\lim}\frac{1}{n}\log s_n(T,\varepsilon)$,
$r_n(T,\varepsilon)=\min\{|F|:F\text{ is }(n,\varepsilon)\text{-dense}\}$, $r(T,\varepsilon)=\overline{\lim}\frac{1}{n}\log r_n(T,\varepsilon)$

Theorem

Let (X,d) compact metric, $T:X\longrightarrow X$ continuous. Then
$$h_{top}(T)=\lim_{\varepsilon\downarrow 0}s(T,\varepsilon)=\lim_{\varepsilon\downarrow 0}r(T,\varepsilon)$$

Theorem (Variational principle)

Let (X,d) compact metric, $T:X\longrightarrow X$ continuous. Then
$$h_{top}(T)=\sup\{h_\mu(T):\mu\in\mathcal{M}_1^T(X)\}$$

Remark:

One can show that, if the entropy map $\mu\longmapsto h_\mu(T)$ is upper semicontinuous, i.e., if $\mu_n\longrightarrow\mu\Rightarrow h_\mu(T)\ge\overline{\lim}\,h_{\mu_n}(T)$, then the sup is attained, i.e., there is $\mu\in\mathcal{M}_1^T(X)$ such that $h_\mu(T)=h_{top}(T)$.

<u>Application towards Littlewood's conjecture</u> (Einsiedler-Katok-Lindenstrauss)

Conjecture (Littlewood ~1930)

$\forall\alpha,\beta\in\mathbb{R}$
$$\liminf_{n\to\infty} n\|n\alpha\|\|n\beta\|=0,\qquad (\star)$$
where $\|\omega\|=\min_{n\in\mathbb{Z}}|\omega-n|$.

How do we relate this to dynamics?

Let $\mathcal{L}=\{\Lambda<\mathbb{R}^3:\Lambda\text{ discrete}, \operatorname{span}_{\mathbb{R}}\Lambda=\mathbb{R}^3\}$. $\mathcal{L}$ is the space of lattices in $\mathbb{R}^3$. One can show that $\Lambda<\mathbb{R}^3$ is a lattice iff there is a basis $\underline{b}\in(\mathbb{R}^3)^3$ of $\mathbb{R}^3$ s.t.
$$\Lambda=\operatorname{span}_{\mathbb{Z}}(\underline{b}).$$

We let $\operatorname{covol}(\Lambda)=|\det(\underline{b})|=\operatorname{vol}\left(\Lambda\backslash\mathbb{R}^3\right)$ and
$$\mathcal{L}^1=\{\Lambda\in\mathcal{L}:\operatorname{covol}(\Lambda)=1\}.$$

Then every element $\Lambda\in\mathcal{L}^1$ is of the form $\Lambda=\mathbb{Z}^3g$ for some $g\in SL_3(\mathbb{R})$ and, letting $G=SL_3(\mathbb{R})$ act on $\mathcal{L}^1$ via $(g,\Lambda)\mapsto\Lambda g^{-1}$, we obtain $\mathcal{L}^1\cong SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})$.

This equips $\mathcal{L}^1$ with a locally compact σ-compact metrizable topology in which bounded sets have compact closure.

For every semigroup $H<SL_3(\mathbb{R})$ we obtain a (multiparameter) dynamical system (X,H), where $X=SL_3(\mathbb{Z})\backslash G$ and $H\curvearrowright X$ by $(g,x)\mapsto xg^{-1}$. Note that H acts by homeomorphisms.

Notation

In what follows

- $\Gamma = SL_3(\mathbb{Z})$
- $A = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 a_2 a_3 = 1 \right\} < SL_3(\mathbb{R})$
- $A^+ = \left\{ a(s,t) = \begin{pmatrix} e^{s+t} & 0 & 0 \\ 0 & e^{-s} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} : s,t \geq 0 \right\}$.
- $\forall \alpha, \beta \in \mathbb{R}$ $g_{\alpha,\beta} = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\Lambda_{\alpha,\beta} = \mathbb{Z}^3 g_{\alpha,\beta}$

Proposition

Let $\alpha, \beta \in \mathbb{R}$, then (α,β) satisfies $(\ast)$ if and only if $A^+ . \Lambda_{\alpha,\beta} \subseteq {}_\Gamma\backslash^G$ is unbounded.

The proof uses the following characterization of compact subsets of ${}_\Gamma\backslash G$.

Theorem (Mahler's compactness criterion)

A set $E \subseteq {}_\Gamma\backslash^G$ is bounded if and only if there is $\varepsilon > 0$ s.t.

$$\forall \Lambda \in E \, \forall v \in \Lambda - \{0\} \quad \|v\|_\infty \geq \varepsilon.$$

In what follows, $\lambda_1(\Lambda) = \inf\{\|v\|_\infty : v \in \Lambda - \{0\}\}$.

Proof of the proposition:

Suppose $A^+ . \Lambda_{\alpha,\beta}$ is unbounded, i.e., $\forall \varepsilon > 0 \, \exists a \in A^+ \, \lambda_1(a . \Lambda_{\alpha,\beta}) < \varepsilon$. Let $\varepsilon \in (0, \frac{1}{2})$ and let $a = a(s,t) \in A^+$ such an element. Then $a . \Lambda_{\alpha,\beta} = \mathbb{Z}^3 g_{\alpha,\beta} a^{-1}$, thus

$$\lambda_1(a . \Lambda_{\alpha,\beta}) < \varepsilon \Rightarrow \exists (n,-m,-k) \in \mathbb{Z}^3 - \{0\} \text{ s.t. } \|(n,-m,-k) g_{\alpha\beta} a^{-1}\|_\infty < \varepsilon$$

$$\Rightarrow \max\{|n| e^{-s-t}, |n\alpha - m| e^{s}, |n\beta - k| e^{t}\} < \varepsilon.$$

Since $\varepsilon < \frac{1}{2}$, we have $n \neq 0$ and $\|n\alpha\| = |n\alpha - m|$, $\|n\beta\| = |n\beta - k|$. Hence

$$n \|n\alpha\| \|n\beta\| < \varepsilon^3.$$

The opposite implication works similarly.

Theorem (Einsiedler-Katok-Lindenstrauss '06)

Let $\delta > 0$. Then

$$\Xi_\delta := \{(\alpha,\beta) \in [0,1]^2 : \liminf_{n \to \infty} n \|n\alpha\| \|n\beta\| \geq \delta\}$$

has zero upper box dimension, i.e.,

$\forall \varepsilon > 0 \, \forall r \in (0,1)$ Ξ_δ can be covered by $O_{\delta,\varepsilon}(r^{-\varepsilon})$ boxes of size $r \times r$.

The main ingredient is the following dynamic result.

Theorem (Einsiedler-Katok-Lindenstrauss '06)

Let μ be an A-invariant ergodic (Borel) probability measure on ${}_\Gamma\backslash^G$, i.e., for all Borel sets $B \subseteq {}_\Gamma\backslash^G$ if $a.B = B$ for all $a \in A$, then $\mu(B) \in \{0,1\}$.

Then either

- μ is the unique G-invariant Borel probability measure on ${}_\Gamma\backslash^G$, the so-called Siegel-Haar measure, or
- $\forall a \in A$ $h_\mu(a) = 0$.

Notation

Given $\sigma, \tau, t \geq 0$, denote $a_{\sigma,\tau}(t) = a(\sigma t, \tau t)$, $a_{\sigma,\tau} := a_{\sigma,\tau}(1)$.

In what follows, we use the following crucial theorem.

Theorem

Let (X,d) compact metric, $T : X \longrightarrow X$ a homeomorphism, $\mu' \in \mathcal{M}_1^T(X)$, and

$$\mu' = \int_Z \mu'_z \, d\sigma(z)$$

an ergodic decomposition. Then $h_{\mu'}(T) = \int_Z h_{\mu'_z}(T) \, d\sigma(z)$. In particular, if $h_{top}(T) > 0$, then there exists $\mu' \in \mathcal{M}_1^T(X)$ ergodic such that $h_{\mu'}(T) > 0$.

Proposition

Let $(\alpha,\beta) \in \mathbb{R}^2$ and suppose that $A^+ . \Lambda_{\alpha,\beta}$ is bounded. Let $\sigma, \tau \geq 0$ and

$$Y = Y_{\alpha,\beta,\sigma,\tau} = \overline{\{a_{\sigma,\tau}(t) . \Lambda_{\alpha,\beta} : t \geq 0\}}.$$

Then $h_{top}(a_{\sigma,\tau}|_Y) = 0$.

Proof Suppose $h_{top}(a_{\sigma,\tau}|_Y) > 0$. By the variational principle, there is $\mathcal{M}_1^{a_{\sigma,\tau}}(Y)$ such that $h_\mu(a_{\sigma,\tau}) > 0$. Given $S > 0$ let

$$\mu_S = \frac{1}{S^2} \int_0^S \int_0^S a(s,t)_* \mu \, ds dt.$$

Since A is commutative, μ_S is $a_{\sigma,\tau}$-invariant. Using the commutativity again, one shows that ξ is a generator of $a(s,t) . Y \supseteq \operatorname{supp}(a(s,t)_* \mu)$ if and only if $a(s,t)^{-1} \xi$ is a generator of Y (with respect to the map $a_{\sigma,\tau}$). Hence

$$h_{a(s,t)_* \sigma}(a_{\sigma,\tau}) = h_{a(s,t)_* \sigma}(a_{\sigma,\tau} | \xi) = h_\sigma(a_{\sigma,\tau} | a(s,t)^{-1} \xi) = h_\sigma(a_{\sigma,\tau}),$$

where we use that generators exist.

It follows that

$$\forall S>0 \quad h_{\mu_S}(a_{\sigma,\tau}) = h_\mu(a_{\sigma,\tau}).$$

Since μ_S is supported on the compact metric space $\overline{A^+.\Lambda_{\alpha,\beta}}$, there is $S_n \uparrow \infty$ such that $\mu_{S_n} \longrightarrow \mu_\infty$. Note that μ_∞ is A^+ and, hence, A-invariant. In our setting the entropy map is upper semicontinuous since $a_{\sigma,\tau}$ behaves similarly to an expansive map. Therefore $h_{\mu_\infty}(a_{\sigma,\tau}) \geq \limsup_{n\to\infty} h_{\mu_{S_n}}(a_{\sigma,\tau}) > 0$ and, therefore, $EKL \Rightarrow \mu_\infty$ is Haar on $\Gamma\backslash G$, which is not compactly supported. In particular, $h_{top}(a_{\sigma,\tau}|_Y) = 0$. $\boxtimes$

Proposition

Let (X', d') a compact metric space, $a: \mathbb{R}\times X' \longrightarrow X'$, $(t,x) \longmapsto a(t).x$ a continuous $\mathbb{R}$-action.

Suppose $X'_0 \subseteq X'$ is compact and $a(t)$ invariant, i.e., $a(t).X'_0 \subseteq X'_0$, and suppose that

$$\forall x\in X'_0 \quad h_{top}(a(t)|_{Y_x}) = 0 \qquad Y_x = \overline{\{a(t').x : t' \geq 0\}}.$$

Then $h_{top}(a(t)|_{X'_0}) = 0$.

Proof: Suppose $h_{top}(a(t)|_{X'_0}) > 0$ and let $\mu \in \mathcal{M}_1^{a(t)}(X'_0)$ ergodic such that $h_\mu(a(t)) > 0$.

For μ-a.e. $x \in X'_0$ $\operatorname{supp}\mu = Y_x$, where we use the individual ergodic theorem for flows, which says that (Einsiedler-Ward, Corollary 8.15)

$$\frac{1}{T}\int_0^T a(t')_* \delta_x \, dt' \xrightarrow{T\to\infty} \mu \qquad \mu\text{-a.s. and in } L^1(\mu).$$

Hence $h_{top}(a(t)|_{Y_x}) \geq h_\mu(a(t)) > 0$. $\boxtimes$

Corollary

Let $C \subseteq \Gamma\backslash G$ compact, $X_C = \{\Lambda \in \Gamma\backslash G : A^+.\Lambda \subseteq C\}$. Then

$$\forall \sigma,\tau \geq 0 \quad h_{top}(a_{\sigma,\tau}|_{X_C}) = 0.$$

Proof of progress towards Littlewood's conjecture

We first note that the characterization of exceptions to Littlewood's conjecture implies

$$\liminf_{n\to\infty} n\|n\alpha\|\|n\beta\| \geq \delta \Rightarrow \lambda_1(A^+.\Lambda_{\alpha,\beta}) \geq \delta^{\frac{1}{3}}.$$

In particular $\forall \delta>0 \, \exists C_\delta \subseteq \Gamma\backslash G$ compact with $(\alpha,\beta) \in \Xi_\delta \Rightarrow A^+.\Lambda_{\alpha,\beta} \subseteq C_\delta$.

We have to say a few words about the metric on G and on $\Gamma\backslash G$. We equip G with a left-invariant metric Riemannian metric d_G. The quotient topology on $\Gamma\backslash G$ is then induced by the metric

$$d(\Gamma g, \Gamma h) = \inf_{\gamma\in\Gamma} d_G(\gamma g, h).$$

It isn't hard to show that every $\Lambda \in \Gamma\backslash G$ admits a positive injectivity radius, i.e.,

$$\varrho_\Lambda = \sup\{r>0 : p_\Lambda : B_r^G(\mathbb{1}) \longrightarrow \Gamma\backslash G, g \longmapsto \Lambda g \text{ is injective}\} > 0.$$

Moreover, for any $C \subseteq \Gamma\backslash G$ compact, there is $\varrho_C > 0$ s.t. $\inf\{\varrho_\Lambda : \Lambda \in C\} = \varrho_C$.

Next, we remark that there is $R_0 > 0$ such that

$$\forall g,h \in B_{R_0}(\mathbb{1}) \qquad d_G(g,h) = \|h^{-1}g - \mathbb{1}\|_\infty. \qquad \text{(Einsiedler-Ward 9.12)}$$

It follows that $\exists c_0, r_0 > 0$ s.t. $\forall \Lambda \in C_\delta \, \forall |\alpha|,|\beta| < r_0$

$$d(\Lambda, g_{\alpha,\beta}.\Lambda) \geq c_0 \max(|\alpha|,|\beta|).$$

With this out of the way, let's return to the proof of the theorem. We need to show that Ξ_δ has upper box dimension 0, i.e.,

$$\forall \varepsilon>0 \, \forall r\in(0,1) \quad \Xi_\delta \text{ can be covered by } O_{\delta,\varepsilon}(r^{-\varepsilon}) \text{ boxes of sizes } r\times r.$$

Equivalently, it suffices to show that any r-separated $S \subseteq \Xi_\delta$, i.e.,

$$\forall x,y \in S \quad x\neq y \Rightarrow \|x-y\|_\infty > r,$$

has cardinality $O_{\delta,\varepsilon}(r^{-\varepsilon})$.

Let $S \subseteq \Xi_\delta$ r-separated, $(\alpha,\beta),(\alpha',\beta') \in S$ and note that $\forall n \in \mathbb{N}$

$$a_{1,1}^n.\Lambda_{\alpha,\beta} = a_{1,1}^n(g_{\alpha'-\alpha,\beta'-\beta}.\Lambda_{\alpha',\beta'}) = a_{1,1}^n g_{\alpha'-\alpha,\beta'-\beta} a_{1,1}^{-n}.(a_{1,1}^n.\Lambda_{\alpha',\beta'})$$

$$= g_{e^{3n}(\alpha'-\alpha), e^{3n}(\beta'-\beta)}.\Lambda_{\alpha',\beta'}.$$

Hence, if $r = e^{-3n} r_0$, then $S' := \{\Lambda_{\alpha,\beta} : (\alpha,\beta) \in S\}$ satisfies

$$\forall \Lambda,\Lambda' \in S' \quad \Lambda\neq\Lambda' \Rightarrow \exists\, 0\leq m<n \text{ s.t. } d(a_{1,1}^m.\Lambda, a_{1,1}^m.\Lambda') \geq c_0 r_0,$$

i.e., S' is $(n, c_0 r_0)$-separated. Since $S' \subseteq X_{C_\delta}$ and $h_{top}(a_{1,1}|_{X_{C_\delta}}) = 0$, it follows that

$$|S'| = O_{\delta,\varepsilon}(\exp(\varepsilon n)).$$

End of the course $\boxtimes$