Problem sheet 3

Problem 1

Show that the rotation number of a homeomorphis of $\mathbb{T}$ only depends on the conjugacy class, i.e., suppose that $T_1, T_2, h\colon \mathbb{T} \to \mathbb{T}$ are homeomorphisms such that

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{T_1} & \mathbb{T} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{T} & \xrightarrow[T_2]{} & \mathbb{T} \end{array}$$

commutes, then $\varrho(T_1) = \varrho(T_2)$.

Problem 2

Let $\alpha \in \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$ and consider the circle rotation $R_\alpha\colon \mathbb{T} \to \mathbb{T}$. Let $h\colon \mathbb{T} \to \mathbb{T}$ be a homeomorphism such that

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{R_\alpha} & \mathbb{T} \\ \downarrow{\scriptstyle h} & & {\scriptstyle h}\downarrow \\ \mathbb{T} & \xrightarrow[R_\alpha]{} & \mathbb{T} \end{array}$$

commutes. Show that

$$\exists \beta \in \mathbb{T} \forall x \in \mathbb{T} \quad h(x) = x + \beta.$$

Problem 3

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, and $T_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ be given by

$$\forall x \in \mathbb{T}^2 \quad T_A(x) = xA + \mathbb{Z}^2.$$

Show that the set of periodic points is countably infinite.
Hint: Given $n \in \mathbb{N}$, look at the map $A^n - \mathrm{id}_2$ to deduce countability. To show infinity, note that A^n maps rational points of denominator q to rational points of denominator q. Then use the pigeonhole principle.

Problem 4

Let X be a compact metric space and let $T\colon X \to X$ continuous. We say that T is *topologically weak mixing* if the map

$$\begin{aligned} T \times T\colon X \times X &\longrightarrow X \times X, \\ (x_1, x_2) &\longmapsto \big(T(x_1), T(x_2)\big) \end{aligned}$$

is topologically transitive.

a. Prove that if T is topologically weak mixing, then T is topologically transitive.

b. Suppose that T is a topologically mixing homeomorphism. Show that T is topologically weak mixing.

Problem 5

Let (X, d) be a compact metric space, $T\colon X \to X$ a homeomorphism. A non-zero function $f \in \mathrm{C}(X)$ is an *eigenfunction for* T if there exists $\lambda \in \mathbb{C}$ (the *eigenvalue*) such that

$$\forall x \in X \quad f(Tx) = \lambda f(x).$$

a. Suppose that $f \in \mathrm{C}(X)$ is an eigenfunction with eigenvalue λ. Show that $|\lambda| = 1$.

b. Suppose that T is topologically transitive and suppose that $f \in \mathrm{C}(X)$ is an eigenfunction. Show that $|f|$ is constant.

c. Suppose that T is topologically transitive and suppose that $f, g \in \mathrm{C}(X)$ are eigenfunctions with the same eigenvalue. Show that there is $c \in \mathbb{C}$ such that $f = cg$.

d. Suppose that T is topologically transitive. Show that any $\mathcal{F} \subseteq \mathrm{C}(X)$ consisting of eigenfunctions for pairwise distinct eigenvalues is linearly independent.

e. Suppose that T is topologically transitive. Prove that the eigenvalues of T form a countable subgroup of S^1.

   Hint: $\mathrm{C}(X)$ contains a countable dense set.

f. Let $x_0 \in X$ and define a map

$$\begin{aligned} *\colon \mathcal{O}(x_0) \times \mathcal{O}(x_0) &\longrightarrow \mathcal{O}(x_0), \\ (T^m x_0, T^n x_0) &\longmapsto T^{m+n}(x_0). \end{aligned}$$

Show that $*$ is well-defined and that $\big(\mathcal{O}(x_0), *, x_0\big)$ is a group.

We recall that a compact space G is a *compact group* if there exist $e \in G$ and a pair of continuous maps $m\colon G \times G \to G$ and $i\colon G \to G$ such that (G, m, e) is a group with $g^{-1} = i(g)$ for all $g \in G$. Given a compact group G with a metric d and an element $g \in G$, we denote by (G, R_g) the topological dynamical system defined by

$$\forall g' \in G \quad R_g(g') = gg'.$$

The system (G, R_g) is called a *group rotation*.

f. Suppose that T is topologically transitive and an isometry, i.e.,

$$\forall x, y \in X \quad d(Tx, Ty) = d(x, y).$$

Show that (X, T) is conjugate to a minimal group rotation on a compact metric abelian group.

Problem 6

This exercise uses the Stone–Weierstrass theorem. Let $T\colon \mathbb{T} \to \mathbb{T}$ be a homeomorphism. Show that the following are equivalent.

a. T is topologically conjugate to an irrational rotation.

b. T is topologically transitive and has *topological discrete spectrum*, i.e., the eigenfunction of T span a dense subspace of $\mathrm{C}(\mathbb{T})$.

Hint: Let $\{\lambda_n\colon n \in \mathbb{N}\}$ be the set of eigenvalues of T and for every $n \in \mathbb{N}$ let $f_n \in \mathrm{C}(\mathbb{T})$ non-zero such that $f_n \circ T = \lambda_n f_n$ and $\|f_n\|_\infty = 1$. Show that

$$\overline{\mathrm{d}}(x, y) = \sum_{n \in \mathbb{N}} \frac{|f_n(x) - f_n(y)|}{2^n}$$

defines a metric inducing the usual topology on $\mathbb{T}$. For this you can use the statement that a continuous map from a compact space to a Hausdorff space is a homeomorphism.