Problem sheet 4

Problem 1

Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

a. Draw the corresponding graphs $\mathcal{G}_{A_1}$ and $\mathcal{G}_{A_2}$.

b. Investigate whether the A_i's are irreducible or aperiodic.

c. Investigate whether the corresponding vertex shifts $(X_{\mathcal{G}_{A_i}}, \sigma)$ are topologically transitive or topologically mixing.

Problem 2

Let $\mathcal{G} = (V, E)$ be a finite graph with a cycle. Suppose that $A_{\mathcal{G}}$ is aperiodic. Prove that the vertex shift $(X_{\mathcal{G}}, \sigma)$ is topologically mixing.

Problem 3

Let $\mathcal{C} \subseteq \mathbb{T}$ be the middle-third Cantor set. Show that $(\mathcal{C}, T_3|_{\mathcal{C}})$ is a topological factor of a shift of finite type.

Problem 4

Prove that the odd shift $X_{\text{odd}} \subset \{0,1\}^{\mathbb{Z}}$ is sofic but not of finite type, where X_{odd} is the set of $x \in \{0,1\}^{\mathbb{Z}}$ such that any two 1's in the sequence are separated by an odd number of 0's.

Problem 5

Let A^* denote the language defined by the alphabet $A = \{0,1\}$, i.e.,

$$A^* = \bigcup_{\ell \in \mathbb{N}} A^\ell.$$

Given $w = (i_1, \ldots, i_\ell) \in A^*$, define $w' = (i_1, \ldots, \overline{i_\ell})$, where $\bar{i}$ denotes the negation of i, i.e., $\bar{0} = 1$ and $\bar{1} = 0$. We inductively define a sequence of words by letting $w_1 = (1)$ and $w_{n+1} = (w_n, w_n')$ for $n \geq 1$. Finally, we denote by $w_\infty \in \{0,1\}^{\mathbb{N}}$ the infinite sequence of 0 and 1 obtained in this way.

Define the *substitution map* $s\colon A^* \to A^*$ as follows. We set $s(0) = (1,1)$ and $s(1) = (1,0)$. For $\ell > 1$, we define

$$s(i_1, \dots, i_\ell) = \big(s(i_1), \dots, s(i_\ell)\big).$$

a. Show that $w_n = s^{n-1}(1)$ for all $n \in \mathbb{N}$.

b. Show that the point $w_\infty \in \{0,1\}^{\mathbb{N}}$ is uniformly recurrent for the one-sided shift, i.e., for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that there is an increasing sequence $n_k \to \infty$ satisfying $d(\sigma^{n_i} w_\infty, w_\infty)$ and

$$\forall k \in \mathbb{N} \quad |n_{k+1} - n_k| < N.$$

c. Show that $w_\infty \in \{0,1\}^{\mathbb{N}}$ has non-periodic orbit under the left shift map.

Problem 6

Let A^* denote the language defined by the alphabet $A = \{0,1\}$. Recall that the *word metric* on A^* is defined as follows. Let $v, w \in A^*$ distinct, then $d(v,w) = 2^{-k}$, where

$$k = \min\big\{ \min\{q \in \mathbb{N} \colon v_q \neq w_q\}, \min\{\mathrm{len}(v), \mathrm{len}(w)\} + 1\big\},$$

and $\mathrm{len}(v)$ denotes the length of v, i.e., for any $n \in \mathbb{N}$ and $v \in A^n$ we have $\mathrm{len}(v) = n$. Note that $\{0,1\}^{\mathbb{N}}$ is a completion of A^* with respect to this metric.

Define the substitution map $\zeta\colon A^* \to A^*$ as follows. Let $\zeta(0) = (0,1)$ and $\zeta(1) = (0)$. For $\ell > 1$ we define

$$\zeta(i_1, \dots, i_\ell) = \big(\zeta(i_1), \dots, \zeta(i_\ell)\big).$$

a. Show that $u = \lim_{n\to\infty} \zeta^n(0) \in \{0,1\}^{\mathbb{N}}$ exists, i.e., $\{\zeta^n(0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in A^*.

b. Prove that the Fibonacci sequence $u = 01001010010\ldots$ obtained in (a) is Sturmian, i.e., for any $n \geq 1$ the number of words of length n appearing in u is $n+1$.

Problem 7

Let A be a finite alphabet and $X \subseteq A^{\mathbb{Z}}$ a non-empty subshift. Recall that we denote by $p_X \colon \mathbb{N} \to \mathbb{N}$ the map where $p_X(n)$ denotes the number of distinct words of length n appearing in X, i.e.,

$$\forall n \in \mathbb{N} \quad p_X(n) = \left|\{w \in A^n \colon {}_0[w] \cap X \neq \emptyset\}\right|.$$

a. *(Fekete's lemma)* Let $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that

$$\forall m, n \in \mathbb{N} \quad a_{m+n} \leq a_m + a_n.$$

Show that the limit $\lim_{n\to\infty} \frac{a_n}{n}$ is well-defined in $\mathbb{R} \cup \{-\infty\}$ and that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}.$$

b. Show that the limit

$$h_{\mathrm{top}}(X, \sigma_X) = \lim_{n\in\mathbb{N}} \frac{\log p_X(n)}{n}$$

exists.

c. Let B be a finite alphabet and $Y \subseteq B^{\mathbb{Z}}$ a non-empty subshift. Suppose that (Y, σ_Y) is a topological factor of (X, σ_X), i.e., there exists a continuous surjective map $h \colon X \to Y$ such that

$$h \circ \sigma_X = \sigma_Y \circ h.$$

Show that

$$h_{\mathrm{top}}(Y, \sigma_Y) \leq h_{\mathrm{top}}(X, \sigma_X).$$

Hint: Let $(x_n)_{n\in\mathbb{Z}} \in X$. How many coordinates around x_n do you need to know in order to determine $h(x)_n$?

d. Let A and B be finite alphabets and suppose that $|B| > |A|$. Show that $B^{\mathbb{Z}}$ isn't a topologocial factor of $A^{\mathbb{Z}}$.