Problem sheet 5

Problem 1

In this exercise we want to compute the actual growth rate of the complexity function for a vertex shift. The necessary input is the so-called Perron–Frobenius theorem. To this end, we introduce a bit of notation. A matrix $A \in \mathrm{Mat}_{r,s}(\mathbb{R})$ is *non-negative* if $A_{ij} \geq 0$ for all $1 \leq i \leq r$ and for all $1 \leq j \leq s$. A vector $v \in \mathbb{R}^d$ is *positive* if $v_j > 0$ for all $1 \leq j \leq d$. A non-negative matrix $A \in \mathrm{Mat}_d(\mathbb{R})$ is *irreducible* if for all $1 \leq i, j \leq d$ there exists $n \in \mathbb{N}$ such that $A^n_{ij} \neq 0$. Finally, we denote by $\|\cdot\|_1$ the L^1-norm on $\mathbb{R}^d$ and by $\Delta^{d-1} \subseteq \mathbb{R}^d$ the set of non-negative vectors in the $\|\cdot\|_1$-unit sphere, i.e.,

$$\Delta^{d-1} = \{v \in \mathbb{R}^d : v \text{ non-negative and } \|v\|_1 = 1\}.$$

A subset $\mathcal{P} \subseteq \mathbb{R}^d$ is a *polytope* if it is the convex hull of a finite subset $S \subseteq \mathbb{R}^d$, i.e., if there is $S \subseteq \mathbb{R}^d$ finite such that

$$\mathcal{P} = \left\{ \sum_{i=1}^{r} t_i s_i : r \in \mathbb{N}, s_1, \ldots, s_r \in S, t_1, \ldots, t_r \in [0,1], t_1 + \cdots + t_r = 1 \right\}.$$

In particular, Δ^{d-1} is the polytope defined by the standard basis of $\mathbb{R}^d$.

a. Suppose that $A \in \mathrm{Mat}_d(\mathbb{R})$ is non-zero non-negative. Suppose that $v \in \mathbb{R}^d$ is a positive eigenvector with eigenvalue $\lambda \in \mathbb{C}$. Show that $\lambda \in (0, \infty)$.

b. Suppose that $\mathcal{G} = (V, E)$ is a graph with adjacency matrix $A_{\mathcal{G}}$ and suppose that $A_{\mathcal{G}}$ has a positive eigenvector with eigenvalue $\lambda_{\mathcal{G}}$. Suppose that $X_{\mathcal{G}}$ is non-empty. Show that $h_{\mathrm{top}}(X_{\mathcal{G}}, \sigma) = \log \lambda_{\mathcal{G}}$.

c. Suppose that $A \in \mathrm{Mat}_d(\mathbb{R})$ is non-zero non-negative and suppose that $v, w \in \mathbb{R}^d$ are positive eigenvectors for respective eigenvalues $\lambda, \mu \in \mathbb{C}$. Show that $\lambda = \mu$, i.e., there exists at most one eigenvalue corresponding to a positive eigenvector.

d. Suppose that $A \in \mathrm{Mat}_d(\mathbb{R})$ is non-zero non-negative and suppose that $v \in \mathbb{R}^d$ is a positive eigenvector for eigenvalue λ. Let $\mu \in \mathbb{C}$ be an eigenvalue of A. Show that $|\mu| \leq \lambda$.

   Hint: Look at the growth of the L^1-norm of eigenvectors under application of A.

e. Suppose that $A \in \mathrm{Mat}_d(\mathbb{R})$ is positive (as an element of $\mathbb{R}^{d^2}$) and $v \in \mathbb{R}^d$ is a non-negative eigenvector of A. Show that v is positive.

f. Suppose that $A \in \mathrm{Mat}_d(\mathbb{R})$ is positive. Show that the map

$$\begin{aligned} \overline{A}\colon \Delta^{d-1} &\longrightarrow \Delta^{d-1} \\ v &\longmapsto \frac{Av}{\|Av\|_1} \end{aligned}$$

is well-defined and continuous.

g. An element v in a polytope $\mathcal{P} \subseteq \mathbb{R}^d$ is *extremal* if v can not be expressed as a non-trivial convex combination of elements in $\mathcal{P}$. Given a polytope $\mathcal{P} \subseteq \mathbb{R}^d$, show that the number of extremal points in $\mathcal{P}$ is finite.

h. Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a polytope and let $\mathcal{E}(\mathcal{P})$ the subset of extremal points. Show that $\mathcal{P}$ is the convex hull of $\mathcal{E}(\mathcal{P})$.

i. Suppose that $A \in \mathrm{Mat}_d(\mathbb{R})$ is positive. Given $m \in \mathbb{N} \cup \{0\}$, let $\mathcal{D}_m = \overline{A}^m(\Delta^{d-1})$. Show that $\mathcal{D}_\infty = \bigcap_{m \geq 0} \mathcal{D}_m$ is a polytope and $\overline{A}(\mathcal{D}_\infty) = \mathcal{D}_\infty$.

j. Show that $\overline{A}\big(\mathcal{E}(\mathcal{D}_\infty)\big) = \mathcal{E}(\mathcal{D}_\infty)$.

k. Let $a_1, \dots, a_d > 0$. Show that the matrix

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{d-1} \\ a_d & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is irreducible and admits a positive eigenvector.

l. For $v, w \in \Delta^d$, denote

$$K(v,w) = \sup\big\{\lambda \in \mathbb{R} \colon v - \lambda w \text{ is non-negative.}\big\}$$

and define

$$\begin{aligned} \rho\colon \Delta^d \times \Delta^d &\longrightarrow \mathbb{R} \\ (v,w) &\longmapsto -\log K(v,w) - \log K(w,v). \end{aligned}$$

Show that ρ defines a metric on the interior of Δ^d (which coincides with that positive vectors in Δ^d).

m. Let $A \in \mathrm{Mat}_d(\mathbb{R})$ positive. Show that $\overline{A}$ is a strict contraction on the interior of Δ^d, i.e.,

$$\forall v, w \in \mathrm{int}(\Delta^{d-1}) \quad v \neq w \implies \rho(\overline{A}v, \overline{A}w) < \rho(v, w).$$

n. Let $A \in \mathrm{Mat}_d(\mathbb{R})$ positive. Show that there exists a positive eigenvector and that it is unique up to scaling, i.e., the corresponding eigenvalue has algebraic multiplicity one.

o. Let $A \in \mathrm{Mat}_d(\mathbb{R})$ non-negative and aperiodic. Show that there exists a positive eigenvector and that it is unique up to scaling.

p. Let $A \in \mathrm{Mat}_d(\mathbb{R})$ non-negative and irreducible. Show that there exists a positive eigenvector and that it is unique up to scaling.

Hint: Look at $A + \mathrm{id}_d$.

q. Let $\mathcal{G}$ be an aperiodic graph with associated non-empty two-sided vertex shift X. Given $n \in \mathbb{N}$, let $\mathrm{Per}_n(X, \sigma_X) \subseteq X$ denote the points of period dividing n. Show that

$$h_{\mathrm{top}}(X, \sigma_X) = \lim_{n\to\infty} \frac{\log|\mathrm{Per}_n(X, \sigma_X)|}{n}.$$

Problem 2

Show that the following are probability measure preserving systems.

a. $\big(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathrm{Leb}, R_\alpha\big)$, where $\alpha \in \mathbb{R}$ and $R_\alpha\colon \mathbb{T} \to \mathbb{T}$ is the rotation by α.

b. $\big(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathrm{Leb}, T_p\big)$, where $p \in \mathbb{Z} \setminus \{0\}$ and $T_p\colon \mathbb{T} \to \mathbb{T}$ denotes the multiplication by p.

c. $\big(G, \mathcal{B}(G), m_G, \varphi\big)$, where G is a compact metric group, m_G is the Haar probability measure on G, and $\varphi\colon G \to G$ is a surjective continuous group homomorphism.

d. $\big(X, \mathcal{B}(X), \mu, T\big)$, where $X = [0, 1)$, μ is the measure defined by

$$\forall B \in \mathcal{B}(X) \quad \mu(B) = \frac{1}{\log 2} \int_B \frac{\mathrm{d}x}{1+x}$$

and $T\colon X \to X$ is the map defined by $T(0) = 0$ and

$$\forall x \in X \setminus \{0\} \quad T(x) = \left\{\frac{1}{x}\right\}.$$