Problem sheet 6

Problem 1

Let $T\colon [0,1] \to [0,1]$ be the map defined by $T(x) = x^2$.

a. Show that for every T-invariant Borel probability measure μ

$$\mu\big((0,1)\big) = 0.$$

b. Describe all T-invariant Borel probability measures.

Problem 2

We recall the notion of a *stationary process*: Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{B}_\infty$ be the σ-algebra generated by rectangles in $\mathbb{R}^{\mathbb{Z}}$, i.e., $\mathcal{B}_\infty$ is the smallest σ-algebra on $\mathbb{R}^{\mathbb{Z}}$ containing all preimages of rectangles under the various finite-dimensional projections. The process $(X_n)_{n\in\mathbb{Z}}$ is stationary if for all $x_0, \ldots, x_\ell \in \mathbb{R} \cup \{\infty\}$ and for all $k \in \mathbb{Z}$ we have

$$P(X_k < x_0, \ldots, X_{k+\ell} < x_\ell) = P(X_0 < x_0, \ldots, X_\ell < x_\ell).$$

Let $\alpha \in \mathbb{T}$, $y \in [0,1)$. Define $X_n\colon \mathbb{T} \to \{0,1\}$ by

$$\forall x \in \mathbb{T} \quad X_n(x) = \big(\mathbb{1}_{[0,y)+\mathbb{Z}} \circ R_\alpha^n\big)(x).$$

Show that $(X_n)_{n\in\mathbb{Z}}$ is a stationary process, when $\mathbb{T}$ is equipped with the Lebesgue probability measure.

Problem 3

Let X be a set and $\mathcal{A}$ an algebra over X, i.e.,

a. $\emptyset \in X$,

b. $\forall A \in \mathcal{A}\ X \setminus A \in \mathcal{A}$,

c. $\forall A, B \in \mathcal{A}\ A \cup B \in \mathcal{A}$.

Suppose that $\mu\colon \mathcal{A} \to [0,1]$ is a finitely additive probability, i.e., $\mu(X) = 1$ and

$$\forall A, B \in \mathcal{A} \quad A \cap B = \emptyset \implies \mu(A \cup B) = \mu(A) + \mu(B).$$

Suppose that $T\colon X \to X$ is $\mathcal{A}$-measurabble, i.e., $T^{-1}\mathcal{A} \subseteq \mathcal{A}$, and measure preserving, i.e.,

$$\forall A \in \mathcal{A} \quad \mu(T^{-1}A) = \mu(A).$$

Suppose that $A \in \mathcal{A}$ has positive measure, i.e., $\mu(A) > 0$. Show that there exists $n \in \mathbb{N}$ such that $n \leq \mu(A)^{-1}$ and

$$\mu(A \cap T^{-n}A) > 0.$$

Problem 4

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Show that μ is ergodic if and only if for all $f, g \in \mathrm{L}^2(X, \mu)$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1} \int_X U_T^n(f)\overline{g}\mathrm{d}\mu \overset{N\to\infty}{\longrightarrow} \int_X f\mathrm{d}\mu \int_X \overline{g}\mathrm{d}\mu.$$

Problem 5

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $p \in [1, \infty)$. Show that for any $f \in \mathrm{L}^p(X, \mathcal{B}, \mu)$ the ergodic average

$$A_n f = \frac{1}{N}\sum_{n=0}^{N-1} U_T^n f$$

converges in $\mathrm{L}^p(X, \mathcal{B}, \mu)$ to a T-invariant function as $N \to \infty$.

Problem 6

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Suppose that $A \in \mathcal{B}$ has positive measure. Show that

$$E = \{n \in \mathbb{N}\colon \mu(A \cap T^{-n}A) > 0\}$$

is syndetic, i.e., there exist $s \in \mathbb{N}$ and $k_1, \ldots, k_s \in \mathbb{Z}$ such that

$$\mathbb{N} \subseteq \cup_{i=1}^{s}(E - k_i).$$

Hint: Prove that for all $f \in \mathrm{L}^2(X, \mathcal{B}, \mu)$ we have

$$\frac{1}{N-M}\sum_{n=M}^{N-1} U_T^n f \overset{N-M\to\infty}{\longrightarrow} P_T f,$$

where $P_T\colon \mathrm{L}^2(X, \mathcal{B}, \mu) \to I_T$ denotes the orthogonal projection onto the space of T-invariant functions.