Problem sheet 7

Problem 1

Show that for $n \geq 2$, for any $n \times n$ integral matrix A with determinant ± 1, if A is hyperbolic, i.e., has no eigenvalue of modulus 1, the map

$$\begin{aligned} T\colon \mathbb{T}^n &\longrightarrow \mathbb{T}^n, \\ x &\longmapsto Ax \bmod \mathbb{Z}^n \end{aligned}$$

on $\big(\mathbb{T}^n, \mathcal{B}(\mathbb{T}^n), \mathrm{Leb}\big)$ is ergodic.
Hint: Use Fourier series and note that A has a root of unity as an eigenvalue if and only if A^m has eigenvalue 1 for some $m \in \mathbb{N}$.

Problem 2

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let

$$\begin{aligned} R\colon \mathbb{T} \times \mathbb{T} &\longrightarrow \mathbb{T} \times \mathbb{T}, \\ (x, y) &\longmapsto (x + \alpha, y + \alpha). \end{aligned}$$

Let μ be the Lebesgue measure on $\mathbb{T} \times \mathbb{T}$.

a. Show that for any set of the form $C = A \times B$ with $A, B \in \mathcal{B}(\mathbb{T})$ we have
$$R^{-1}C = C \implies \mu(C) \in \{0, 1\}.$$

b. Show that the transformation R is not ergodic with respect to μ.

Problem 3

a. Given $\alpha_1, \alpha_2 \in \mathbb{R}$, let

$$\begin{aligned} R_{\alpha_i}\colon \mathbb{T} &\longrightarrow \mathbb{T}, \\ x &\longmapsto x + \alpha_i \bmod 1. \end{aligned}$$

Find an arithmetic condition on α_1 and α_2 that is equivalent to the ergodicity of the product system $R_{\alpha_1} \times R_{\alpha_2}\colon \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ with respect to the Lebesgue measure $m_{\mathbb{T}} \times m_{\mathbb{T}}$.

b. Generalize this to characterize ergodicity of $R_{\alpha_1} \times \cdots \times R_{\alpha_n}\colon \mathbb{T}^n \to \mathbb{T}^n$ with respect to the Lebesgue measure on $\mathbb{T}^n$ for $(\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$.

Problem 4

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system. Let $f\colon X \to \mathbb{R}^+$ measurable. Suppose that $\int_X f d\mu = \infty$. Show that

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \to \infty$$

for μ-a.e. $x \in X$.

Problem 5

Let (X, d) be a second countable metric space and let μ be a Borel probability measure on X.

a. Denote by $\mathcal{T}$ the topology on X. Let

$$\mathrm{supp}\mu = \{x \in X \colon \forall V \in \mathcal{T}\ x \in V \implies \mu(V) > 0\}.$$

Show that $\mathrm{supp}\mu$ is measurable and $\mu(\mathrm{supp}\mu) = 1$.

Hint: Use that (X, d) is a *Lindelöf* space. If you are not comfortable with this, assume that (X, d) is compact.

b. Suppose that $\big(X, \mathcal{B}(X), \mu, T\big)$ is ergodic. Show that for μ-a.e. $y \in X$ the set

$$\big\{x \in X \colon \exists (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}} \ \lim_{k\to\infty} T^{n_k}x = y\big\}$$

has full measure.

c. Suppose that $\big(X, \mathcal{B}(X), \mu, T\big)$ is ergodic. Show that for μ-a.e. $x \in X$ the set

$$\big\{y \in X \colon \exists (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}} \ \lim_{k\to\infty} T^{n_k}x = y\big\}$$

contains a full measure subset.

Problem 6

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\mathcal{P} = \{A_n \colon n \in \mathbb{N}\} \subseteq \mathcal{B}$ be a partition of X. Let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{P}$. Show that

$$\forall f \in \mathrm{L}^1(X, \mathcal{B}, \mu) \quad \mathbb{E}(f|\mathcal{A}) = \sum_{\substack{n\in\mathbb{N}\\ \mu(A_n)>0}} \frac{\mathbb{E}(\mathbb{1}_{A_n} f)}{\mu(A_n)} \mathbb{1}_{A_n}.$$

Problem 7

For a number $x \in (0,1)$, we write $x = 0.d_1(x)d_2(x)\dots$ for the decimal expansion of x.

a. Show that for Lebesgue almost every x, the limit of

$$D(x) = \lim_{n\to\infty} \frac{d_1(x) + \cdots + d_n(x)}{n}$$

exists and compute it.

b. Give examples of $x \in (0,1)$ such that $D(x)$ does not exist and such that $D(x) = \frac{1}{2025}$.

Problem 8

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system. Given $A \in \mathcal{B}$, recall that by Poincaré recurrence for μ-a.e. $x \in A$ we have

$$n_A(x) = \inf\left\{n \geq 1\colon T^n(x) \in A\right\} \in \mathbb{N}.$$

a. Suppose that $A \in \mathcal{B}$ and $\mu(A) > 0$. Define μ_A on

$$\mathcal{B} \cap A = \{B \cap A\colon B \in \mathcal{B}\}$$

by

$$\forall B \in \mathcal{B} \cap A \quad \mu_A(B) = \frac{\mu(B)}{\mu(A)}$$

Define the *induced* transformation

$$\begin{aligned} T_A\colon A &\longrightarrow A, \\ x &\longmapsto T^{n_A(x)}(x) \end{aligned}$$

for μ-a.e. $x \in A$. Prove that T_A is an ergodic measure preserving transformation with respect to μ_A.

b. (Kac's Lemma) Suppose $A \in \mathcal{B}$ and $\mu(A) > 0$. Prove that

$$\int_A n_A d\mu = 1.$$