Problem sheet 8

Problem 1

Let (X, d) be a compact metric space and $\mathcal{B} = \mathcal{B}(X)$ the Borel σ-algebra on X. Let $T\colon X \to X$ measurable.

a. Let $\mathcal{M}^T(X)$ be the set of T-invariant probability measures. Show that $\mathcal{M}^T(X)$ is convex.

b. Let $\mu \in \mathcal{M}^T(X)$ and suppose that μ is an extreme point, i.e., for all distinct $\mu_1, \mu_2 \in \mathcal{M}^T(X)$ and for all $t \in [0,1]$ we have

$$\mu = t\mu_1 + (1-t)\mu_2 \implies t \in \{0,1\}.$$

Show that μ is ergodic.

Hint: Argue by contradiction, i.e., suppose that μ admits an invariant set $B \in \mathcal{B}$ such that $\mu(B)\bigl(1 - \mu(B)\bigr) \neq 0$.

c. Let $\mu \in \mathcal{M}^T(X)$. Suppose that μ is ergodic. Show that μ is an extreme point.

Hint: Suppose that $t \in (0,1)$ and $\mu_1, \mu_2 \in \mathcal{M}^T(X)$ are distinct and

$$\mu = t\mu_1 + (1-t)\mu_2.$$

Let $f \in \mathrm{L}^1(X, \mathcal{B}, \mu)$ be the Radon–Nikodym derivative of μ_1 with respect to μ and $B = \{f < 1\}$. Show that $B \in \mathcal{E}_{T,\mu}$.

Problem 2

Let $p \in \mathbb{N} \setminus \{1\}$, $A = \{0, \ldots, p-1\}$, $X = A^{\mathbb{N}}$, $\mathcal{B} = \mathcal{B}(X)$.

a. Let μ be a shift-invariant ergodic probability measure on $(X, \mathcal{B})$. Show that there exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ of shift-invariant probability measures such that the following are true.

- For every $n \in \mathbb{N}$, ν_n assigns full measure to the orbit of a single periodic point.
- $\lim_{n\to\infty} \nu_n = \mu$ in the weak-$*$ topology, i.e.,

$$\forall f \in \mathrm{C}(X)\colon \lim_{n\to\infty} \nu_n(f) = \mu(f).$$

b. Let $m \in \mathbb{N}$ and $\mu_1, \dots, \mu_m$ be shift-invariant ergodic probability measures on $(X, \mathcal{B})$. Let $(t_1, \dots, t_m) \in \mathbb{R}^m$ be a probability vector. Show that there exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ of shift-invariant probability measures such that the following are true.

- For every $n \in \mathbb{N}$, ν_n assigns full measure to the orbit of a single periodic point.
- $\lim_{n\to\infty} \nu_n = \sum_{k=1}^m t_k \mu_k$ in the weak-$*$ topology.

Problem 3

Let (X, d) be a compact metric space and $T\colon X \to X$ be a continuous map.

a. Two T-invariant Borel probability measures μ, ν are called *(mutually) singular* if there exists $A \in \mathcal{B}(X)$ such that

$$\mu(A) = 1 \quad \text{and} \quad \nu(A) = 0.$$

Show that if $\mu \neq \nu$ are ergodic, then they are mutually singular.

b. Let μ, ν be T-invariant Borel probability measures. Suppose that there exists $C > 0$ so that

$$\forall f \in \mathrm{C}(X) \quad f \geq 0 \implies \int_X f \mathrm{d}\nu \leq C \int_X f \mathrm{d}\mu$$

Show that ν is absolutely continuous with respect to μ.

Hint: Extend the estimate to indicator functions of open sets.

c. Suppose that μ is an ergodic T-invariant Borel probabilty measure on X and suppose that for every $x \in X$, there exists $C(x) > 0$ so that

$$\forall f \in \mathrm{C}(X) \quad f \geq 0 \implies \limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^n x) \leq C(x) \int_X f \mathrm{d}\mu.$$

Show that T is uniquely ergodic, i.e., μ is the unique T-invariant Borel probability measure on X.

Problem 4

Suppose that (X, d) is a compact metric space. Let $\mathcal{B} = \mathcal{B}(X)$, μ a Borel probability measure on X. Suppose that $T\colon X \to X$ is measurable and surjective and suppose that $T_*\mu = \mu$. Define

$$\tilde{X} = \big\{(x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}} \colon \forall n \in \mathbb{Z} \colon x_{n+1} = T(x_n)\big\}.$$

Let $\tilde{\mathcal{B}}$ be the smallest σ-algebra containing all sets of the form

$$_n[E] = \{(x_n)_{n\in\mathbb{Z}} \in \tilde{X} \colon x_n \in E\} \quad (n \leq 0, E \in \mathcal{B}).$$

a. Show that there exists a unique probability measure $\tilde{\mu}$ on $(\tilde{X}, \tilde{\mathcal{B}})$ such that

$$\forall n \leq 0 \forall E_n \in T^n\mathcal{B} \quad \tilde{\mu}({}_n[E_n]) = \mu(E_n).$$

Hint: Note that for $N \in \mathbb{N}$ and for every choice of $E_k \in T^{-k}\mathcal{B}$ $(0 \leq k \leq N)$ we have

$$\bigcap_{k=0}^{N} {}_{-k}[E_k] = \left\{(x_n)_{n\in\mathbb{Z}} \in \tilde{X} \colon x_{-N} \in \bigcap_{k=0}^{N} T^{-(N-k)} E_{-k}.\right\}$$

b. Let $\tilde{T} \colon \tilde{X} \to \tilde{X}$ denote the left-shift. Show that $\tilde{T}_*\tilde{\mu} = \tilde{\mu}$.

c. Show that $(\tilde{X}, \mathcal{B}, \tilde{\mu}, \tilde{T})$ is an invertible dynamical system and $(X, \mathcal{B}, \mu, T)$ is a factor.

d. Show that $(\tilde{X}, \mathcal{V}, \tilde{\mu}, \tilde{T})$ is ergodic if and only if $(X, \mathcal{B}, \mu, T)$ is ergodic.

e. Suppose that $(Y, \mathcal{C}, \nu, S)$ is an invertible dynamical system and suppose that $(X, \mathcal{B}, \mu, T)$ is a factor. Show that $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu}, \tilde{T})$ is a factor of $(Y, \mathcal{C}, \nu, S)$ and deduce that this property characterizes $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu}, \tilde{T})$ uniquely up to isomorphism.

f. Let $p \in \mathbb{N}$, $A = \{0, \ldots, p-1\}$, $X = A^{\mathbb{N}}$, $\mathcal{B} = \mathcal{B}(X)$. Let $(p^{(\ell)})_{\ell\in\mathbb{N}\cup\{0\}}$ be a sequence of maps $p^{(\ell)} \colon A^{\ell+1} \to [0,1]$ satisfying

$$p^{(0)}(0) + \cdots + p^{(0)}(p-1) = 1$$

and

$$\forall \ell \in \mathbb{N} \forall (a_0, \ldots, a_\ell) \quad p^{(\ell)}(a_0, \ldots, a_\ell) = \sum_{a\in A} p^{(\ell+1)}(a_0, \ldots, a_\ell, a).$$

Let $\mu_{\mathbb{N}}$ be the probability measure on $(X, \mathcal{B})$ uniquely determined by the requirement that for all $q \in \mathbb{N}$, $\ell \in \mathbb{N} \cup \{0\}$, $\underline{a} = (a_0, \ldots, a_\ell) \in A^{\ell+1}$ we have

$$\mu_{\mathbb{N}}\big(\{(x_n)_{n\in\mathbb{N}} \in X \colon \forall 0 \leq k \leq \ell \, x_{q+k} = a_k\}\big) = p^{(\ell)}(\underline{a}).$$

Similarly, one defines the Borel probability measure $\mu_{\mathbb{Z}}$ on $\big(A^{\mathbb{Z}}, \mathcal{B}(A^{\mathbb{Z}})\big)$ by letting q vary in $\mathbb{Z}$. Let $\sigma_+ \colon X \to X$ the left-shift on X. Show that

$$\big(A^{\mathbb{Z}}, \mathcal{B}(A^{\mathbb{Z}}), \mu_{\mathbb{Z}}, \sigma\big) = \big(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu}_{\mathbb{N}}, \widetilde{\sigma_+}\big),$$

where σ denotes the left-shift on $A^{\mathbb{Z}}$.