Problem sheet 9

Problem 1

Let $\left(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu\right)$ and $\left(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \nu\right)$ be irreducible $(\mathbf{p}, P)$- and $(\mathbf{q}, P)$-Markov chains respectively. Suppose that $\mathbf{p}P = \mathbf{p}$.

a. Show that $\mathrm{L}^1(S^{\mathbb{N}_0}, \mu) \subseteq \mathrm{L}^1(S^{\mathbb{N}_0}, \nu)$.

b. Let $f \in \mathrm{L}^1(S^{\mathbb{N}_0}, \mu)$. Show that

$$\frac{1}{N}\sum_{n=0}^{N-1} f \circ \sigma^n \overset{N\to\infty}{\longrightarrow} \int f \mathrm{d}\mu$$

in $\mathrm{L}^1(S^{\mathbb{N}_0}, \nu)$.

Problem 2

Let G be a finite group and suppose that $\pi \in [0,1]^G$ is a probability vector on G. Define $P \in [0,1]^{G\times G}$ by

$$\forall g, h \in G \quad P_{g,h} = \pi(g^{-1}h).$$

a. Show that P is a stochastic matrix, i.e.,

$$\forall g \in G \quad \sum_{h\in G} P_{g,h} = 1.$$

b. Show that the uniform distribution $\mathbf{p} = \frac{1}{|G|}\mathbb{1}_G$ is a stationary distribution for P.

c. Let $\Sigma = \{g \in G \colon \pi(g) > 0\}$. Show that P is irreducible if and only if the set Σ generates G, i.e., every element in G is a finite product of elements in Σ.

d. Let $\Sigma = \{g \in G \colon \pi(g) > 0\}$, $H < G$ the subgroup generated by Σ, and fix an element $g \in G$. Let $\left(G^{\mathbb{N}_0}, \mathcal{B}(G^{\mathbb{N}_0}), \mu_g\right)$ be the (δ_g, P)-Markov chain determined by the Dirac mass at g. Show that

$$\forall f \in \mathbb{C}^G \quad \frac{1}{N}\sum_{n=0}^{N-1} f \circ \pi_n \overset{N\to\infty}{\longrightarrow} \int_H f(gh)\mathrm{d}m_H(h) \quad \mu_g\text{-a.s.},$$

where m_H is the Haar probability measure on H.

Problem 3

(The Ehrenfests' urn model) Let $N \in \mathbb{N}$. Suppose we are given $2N$ (numbered and distinguishable) particles and two urns denoted A and B. Define the state space $S = \{0, \ldots, 2N\}$. We observe the system consisting of the two urns and the particles distributed among them. The state variable π_n ($n \in \mathbb{N}_0$) equals the number of particles in urn A at time n. The evolution of the system is described as follows. At any given time $n \in \mathbb{N}_0$, a particle is chosen uniformly among the $2N$ particles and moved to the urn not containing it. That is, the evolution is given by a transition matrix $P = (p_{k,\ell})_{k,\ell \in S}$ with

$$p_{k,\ell} = \begin{cases} \frac{k}{2N} & \text{if } \ell = k-1, \\ 1 - \frac{k}{2N} & \text{if } \ell = k+1. \end{cases}$$

a. Show that P is irreducible.

b. Let $\mathbf{p} \in [0,1]^S$ be the unique stationary distribution for P. Show that

$$\forall k \in \{-N, \ldots, N\} \quad \mathbf{p}(N+k) = \frac{(2N)!}{(N+k)!(N-k)!} 2^{-2N}.$$

Hint: Use induction on $N + k$.
Remark: Note that

$$\mathbf{p}(N+k) = P(X_N = N+k)$$

for any $X_N \sim \mathrm{Bin}(2N, 1/2)$.

c. In this subexercise, we derive the Central Limit Theorem for (an even number of) Bernoulli trials. This is an immensely important result in probability theory and has close connections to dynamics. However, we are currently lacking both the necessary probabilistic and dynamical methods for a dynamic treatment. If short on time I suggest you **skip** this subexercise and use the conclusion of this result as is.

Given $x > 0$, let

$$P_N(x) = \sum_{|k| < x\sqrt{N/2}} \mathbf{p}(N+k).$$

Show that

$$\lim_{N \to \infty} P_N(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t.$$

To this end, you can proceed as follows.

(i) Show that for $0 \leq k \leq N$ we have

$$\mathbf{p}(N+k) = \mathbf{p}(n) D_{k,N},$$

where

$$D_{k,N} = \left(\prod_{\ell=0}^{k-1} \left(1 + \frac{k}{N-\ell} \right) \right)^{-1}.$$

(ii) Define $\varepsilon_{k,N} > 0$ by

$$\log D_{k,N} = -(1+\varepsilon_{k,N}) \sum_{\ell=0}^{k-1} \frac{k}{n-\ell}.$$

Show that for all $x > 0$

$$\lim_{N\to\infty} \max_{0 \leq k < x\sqrt{N/2}} |\varepsilon_{k,N}| = 0.$$

(iii) Deduce that

$$\log D_{k,N} = -(1+\varepsilon'_{k,N}) \frac{k^2}{N}$$

for some $\varepsilon'_{k,N}$ satisfying

$$\lim_{N\to\infty} \max\{0 \leq k < x\sqrt{N/2}\} |\varepsilon'_{k,N}| = 0.$$

(iv) Use Stirling's approximation to deduce that

$$\mathbf{p}(N) = \frac{1}{\sqrt{\pi N}}(1+\delta_N)$$

for a sequence $(\delta_N)_{N\in\mathbb{N}}$ satisfying $\lim_{N\to\infty} \delta_N = 0$.

(v) Deduce that

$$P_N(x) = \frac{1+\delta'_N}{\sqrt{\pi N}} \sum_{|k|<x\sqrt{N/2}} e^{-\frac{k^2}{N}}$$

for some $\delta'_N \to 0$ as $N \to \infty$.

(vi) Set $t_{k,N} = k\sqrt{\frac{2}{N}}$ and $\Delta_N t = \sqrt{\frac{2}{N}}$. Show that

$$P_N(x) = \frac{1+\delta'_N}{\sqrt{2\pi}} \sum_{-x<t_{k,N}<x} e^{-\frac{t_{k,N}^2}{2}} \Delta_N t$$

and deduce that

$$\lim_{N\to\infty} P_N(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-\frac{t^2}{2}} \,\mathrm{d}t.$$

d. Let $x > 0$ and let τ_x denote the stopping time given by

$$\tau_x = \inf\{n \in \mathbb{N} \colon \pi_n > N + x\sqrt{N/2}\}.$$

Suppose that $\left(S^{\mathbb{N}_0}, \mathcal{B}(S^{\mathbb{N}_0}), \mu\right)$ is the stationary $(\mathbf{p}, P)$-Markov chain. Show that

$$\mathbb{E}_\mu(\tau_x) \gg x e^{\frac{x^2}{2}}.$$

Hint: Use Mills' ratio.

Problem 4

Let $X = \{z \in \mathbb{C} \colon |z - i| = 1\}$. We define the stereographic projection π from X to the real axis by continuing the line from $2i$ through a unique point on $X \setminus \{2i\}$ until it meets the real axis.

The "North-South" map $T \colon X \to X$ is defined by

$$T(x) = \begin{cases} 2i & \text{if } z = 2i, \\ \pi^{-1}\left(\frac{\pi(x)}{2}\right) & \text{if } z \neq 2i. \end{cases}$$

Describe all T-invariant measures on X and find all ergodic ones.

Problem 5

Let (X, d) be a compact metric space and $T \colon X \to X$ continuous. Suppose that the system is is uniquely ergodic, i.e., there exists exactly one Borel probability measure μ on X such that $T_*\mu = \mu$. Prove that for any $f \in \mathrm{C}(X)$ there exists a constant $C_f \in \mathbb{C}$ so that

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \overset{N\to\infty}{\longrightarrow} C_f$$

uniformly in $x \in X$.
Hint: Prove this by contradiction – assuming otherwise, construct a T-invariant ergodic measure different from μ.

Problem 6

Consider the doubling map

$$\begin{aligned} T_2 \colon \mathbb{T} &\longrightarrow \mathbb{T}, \\ x &\longmapsto 2x \mod 1 \end{aligned}$$

and equip $\mathbb{T}$ with the Lebesgue measure $m_{\mathbb{T}}$.

a. Construct a point that is generic for $m_{\mathbb{T}}$.

b. Construct a point that is generic for an ergodic T_2-invariant Borel probabiliy measure other than $m_{\mathbb{T}}$.

c. Construct a point that is generic for a non-ergodic T_2-invariant measure.

d. Construct a point that is *not* generic for any T_2-invariant measure.