Problem sheet 1

Problem 1

Let

$$\begin{aligned} T\colon [0,1] &\longrightarrow [0,1], \\ x &\longmapsto \begin{cases} 0 & \text{if } x \in \{0,1\}, \\ nx-1 & \text{if there is } n \in \mathbb{N} \text{ so that } x \in [\frac{1}{n}, \frac{1}{n-1}). \end{cases} \end{aligned}$$

a. Let $x \in [0,1]$. Show that

$$\exists \ell \in \mathbb{N} \quad T^\ell(x) = 0 \iff x \in [0,1] \cap \mathbb{Q}.$$

b. Show that $e = \sum_{k=0}^\infty \frac{1}{k!}$ is irrational.

Problem 2

Let $T_p\colon \mathbb{T} \to \mathbb{T}$ be the $\times p$-map.

a. Show that there exists $x \in \mathbb{T}$ with $\omega^+(x)$ uncountable but not $\mathbb{T}$, where

$$\omega^+(x) = \big\{ y \in \mathbb{T} \colon \exists (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}} \text{ unbounded, } y = \lim_{k\to\infty} T_p^{n_k}(x) \big\}.$$

b. Let $x_0 = \sum_{k=1}^\infty \frac{1}{p^{k!}}$ show that $\omega^+(x_0)$ is countable but not finite.

Problem 3

Let A be a 2×2 real matrix with eigenvalues $\lambda \in (1,\infty)$ and $\mu \in (0,1)$. Consider the map

$$\begin{aligned} T\colon S^1 &\longrightarrow S^1, \\ x &\longmapsto \frac{Ax}{\|Ax\|_2}. \end{aligned}$$

a. Show that T has exactly four fixed points.

b. Show that $T^n x = \frac{A^n x}{\|A^n x\|_2}$.

c. Show that for every $x \in S^1$, the sequence $T^n x$ converges to a fixed point of T.

Problem 4

Let $p \geq 2$ and

$$X = \prod_{i=1}^{\infty} \{0, ..., p-1\}.$$

Define

$$\begin{aligned} N\colon X \times X &\longrightarrow \mathbb{N} \cup \infty, \\ (x,y) &\longmapsto \inf\{n \in \mathbb{N}\colon x_n \neq y_n\} \end{aligned}$$

and

$$\begin{aligned} d\colon X \times X &\longrightarrow [0, \infty), \\ (x,y) &\longmapsto \begin{cases} 0 & \text{if } x = y, \\ \left(\frac{1}{2}\right)^{N(x,y)} & \text{else.} \end{cases} \end{aligned}$$

Then d defines a metric on X and we equip X with the topology generated by the open balls with respect to the metric d.

Let $\sigma\colon X \to X$ be the left-shift, i.e.,

$$\forall x \in X \forall n \in \mathbb{N} \quad \sigma(x)_n = x_{n+1}.$$

a. Let $(x^{(\ell)})_{\ell \in \mathbb{N}} \in X^{\mathbb{N}}$. Show that $x^{(\ell)} \to x^{(1)}$ as $\ell \to \infty$ if and only if

$$\forall n \in \mathbb{N} \exists \ell_0 \in \mathbb{N} \forall \ell \geq \ell_0 \quad x_n^{(\ell)} = x_n^{(1)}.$$

b. Show that X is sequentially compact: Every sequence $(x^{(\ell)})_{\ell \in \mathbb{N}} \in X^{\mathbb{N}}$ has a converging subsequence.

c. Determine the number of σ-periodic points of period n.

d. Show that the set of σ-periodic points is dense in X.