Problem sheet 10

Problem 1

(Furstenberg) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Show that T is weak-mixing if and only if for all $A, B, C \in \mathcal{B}$ we have that

$$\mu(A)\mu(B)\mu(C) > 0 \implies \exists n \in \mathbb{N} \mu(T^{-n}A \cap B)\mu(T^{-n}A \cap C) > 0.$$

Hint: To deduce that the formal statement implies weak-mixing, one coulde argue as follows. First, note that the system necessarily is ergodic and suppose for sake of contradiction that $f \in \mathrm{L}^2(X)$ is an eigenfunction for eigenvalue $\lambda \in \mathbb{S}^1 \setminus \{1\}$. Show that there exist disjoint intervals $I, J \subseteq \mathbb{S}^1$ and $r > 0$ such that

$$\mu\big(f^{-1}(rI)\big)\mu\big(f^{-1}(rJ)\big)$$

Problem 2

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system.

a. Suppose that for all $A, B \in \mathcal{B}$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

Show that

$$\forall A \in \mathcal{B} \quad \mu(A) \in \{0, 1\}.$$

b. Suppose that for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\forall A, B \in \mathcal{B} \forall n \geq n_\varepsilon \quad |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| < \varepsilon.$$

Show that

$$\forall A \in \mathcal{B} \quad \mu(A) \in \{0, 1\}.$$

Problem 3

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, $\theta \in \mathbb{R} \setminus \mathbb{Z}$. Denote

$$U \colon \mathrm{L}^2(X, \mu) \longrightarrow \mathrm{L}^2(X, \mu), \quad f \longmapsto e^{2\pi \mathrm{i}\theta}(f \circ T)$$

and let

$$V = \{f \in \mathrm{L}^2(X, \mu) \colon f \circ T = e^{-2\pi \mathrm{i}\theta} f\},$$

i.e., V is the subspace of U-fixed vectors. Denote by $P \colon \mathrm{L}^2(X, \mu) \to V$ the orthogonal projection onto V.

a. Show that for all $f \in \mathrm{L}^2(X, \mu)$ we have

$$\frac{1}{N} \sum_{n=0}^{N-1} U^n f \overset{N\to\infty}{\longrightarrow} Pf$$

in $\mathrm{L}^2(X, \mu)$ as $N \to \infty$.

b. Suppose that T is weak-mixing. Show that

$$\frac{1}{N} \sum_{n=0}^{N-1} U^n f \overset{N\to\infty}{\longrightarrow} 0$$

in $\mathrm{L}^2(X, \mu)$ as $N \to \infty$.

Problem 4

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Show that the irrational rotation $R_\alpha \colon \mathbb{T} \to \mathbb{T}$ isn't weak-mixing.

Problem 5

Let $p \in \mathbb{N} \setminus \{1\}$. Show that the times-$p$ map $T_p \colon \mathbb{T} \to \mathbb{T}$ is mixing.

Problem 6

Let $n \in \mathbb{N}$ and let $A \in \mathrm{GL}_n(\mathbb{Z})$ hyperbolic, i.e., suppose that A doesn't have eigenvalues of modulus 1. Show that the induced toral automorphism $T_A \colon \mathbb{T}^n \to \mathbb{T}^n$ is mixing with respect to Lebesgue measure.