Problem sheet 11

Problem 1

Let (X,d) be a compact metric space continuous, and $\{f_n\colon n\in\mathbb{N}\}\subseteq \mathrm{C}(X)\setminus\{0\}$ dense. Show that the metric

$$\begin{aligned} D\colon \mathcal{M}_1(X)^2 &\longrightarrow [0,\infty),\\ (\mu,\nu) &\longmapsto \sum_{n\in\mathbb{N}} \frac{|\mu(f_n)-\nu(f_n)|}{2^n\|f_n\|_\infty} \end{aligned}$$

induces the weak-$*$ topology on $\mathcal{M}_1(X)$.

Problem 2

Let (X,d) be a compact metric space, $T\colon X\to X$ continuous, and $\mu\in\mathcal{M}_1^T(X)$.

a. Show that μ is ergodic if and only if

$$\forall \nu\in\mathcal{M}_1(X)\quad \nu\ll\mu \implies \frac{1}{N}\sum_{n=0}^{N-1} {T^n}_*\nu \overset{N\to\infty}{\longrightarrow} \mu.$$

b. Show that μ is mixing if and only if

$$\forall \nu\in\mathcal{M}_1(X)\quad \nu\ll\mu \implies {T^n}_*\nu \overset{n\to\infty}{\longrightarrow} \mu.$$

c. Show that μ is weak mixing if and only if there exists $J\subseteq\mathbb{N}$ such that

$$\forall \nu\in\mathcal{M}_1(X)\quad \nu\ll\mu \implies {T^n}_*\nu \underset{n\notin J}{\overset{n\to\infty}{\longrightarrow}} \mu.$$

Problem 3

Let (X,d) a compact metric space and $T\colon X\to X$ continuous and $\mu\in\mathcal{M}_1^T(X)$. Show that μ is ergodic if and only if there exists $Y\in\mathcal{B}(X)$ such that $\mu(Y)=1$ and

$$\forall x\in Y\quad \frac{1}{N}\sum_{n=0}^{N-1}\delta_{T^n x} \overset{N\to\infty}{\longrightarrow} \mu.$$

Problem 4

Let (X,d) a compact metric space and suppose that $T\colon X\to X$ and $\varphi\colon X\to\mathbb{T}$ are continuous maps. Denote by m the normalized (Lebesgue) Haar measure on $\mathbb{T}$. We let $\pi_X\colon X\times\mathbb{T}\to X$ and $\pi_{\mathbb{T}}\colon X\times\mathbb{T}\to\mathbb{T}$ denote the coordinate projections. Define

$$\begin{aligned}\hat{T}\colon X\times\mathbb{T} &\longrightarrow X\times\mathbb{T},\\ (x,t) &\longmapsto \big(Tx, t+\varphi(x)\big);\end{aligned}$$

the map $\hat{T}$ is called the *skew-product* map

a. Suppose that $\mu\in\mathcal{M}_1^T(X)$. Show that $\hat{\mu}=\mu\otimes m\in\mathcal{M}_1^{\hat{T}}(X\times\mathbb{T})$.

b. Given $t\in\mathbb{T}$, let $\tau_t\colon\mathbb{T}\to\mathbb{T}$ denote translation by t, i.e., $\tau_t(s)=s+t$ for all $s\in\mathbb{T}$.

   Let $t\in\mathbb{T}$ and $\nu\in\mathcal{M}_1^{\hat{T}}(X\times\mathbb{T})$. Show that $\nu_t=\tau_{t*}\nu\in\mathcal{M}_1^{\hat{T}}(X\times\mathbb{T})$.

c. Given $\nu\in\mathcal{M}_1^{\hat{T}}(X\times\mathbb{T})$, define $\overline{\nu}=\int_{\mathbb{T}}\nu_t\mathrm{d}m(t)$ by

$$\forall f\in\mathrm{C}(X)\quad \overline{\nu}(f)=\int_{\mathbb{T}}\nu_t(f)\mathrm{d}m(t).$$

   Show that $\overline{\nu}=\{\pi_X\}_*\nu\otimes m$.

d. Suppose T is uniquely ergodic and suppose that $\hat{\mu}$ is $\hat{T}$-ergodic. Show that $\hat{T}$ is uniquely ergodic.

Problem 5

Let $\alpha\in\mathbb{R}\setminus\mathbb{Q}$. Define

$$\begin{aligned}T_\alpha\colon\mathbb{T}^2 &\longrightarrow\mathbb{T}^2,\\ (x,y) &\longmapsto (x+\alpha, y+x).\end{aligned}$$

a. Show that the Haar measure on $\mathbb{T}^2$ is T_α-ergodic.

   Hint: Recall that any $f\in\mathrm{L}^2(\mathbb{T}^2)$ admits a unique representation

$$f=\sum_{\mathbf{n}\in\mathbb{Z}^2}c_{\mathbf{n}}e_{\mathbf{n}},$$

   where $c_{\mathbf{n}}\in\ell^2(\mathbb{Z}^2)$ and

$$\forall\mathbf{x}\in\mathbb{R}^2\forall\mathbf{n}\in\mathbb{Z}^2\quad e_{\mathbf{n}}(\mathbf{x}+\mathbb{Z}^2)=\exp\big(2\pi\mathrm{i}\langle\mathbf{n},\mathbf{x}\rangle\big).$$

b. Show that $\{n^2\alpha \colon n \in \mathbb{N}\}$ equidistributes in $\mathbb{T}$, i.e.,

$$\forall f \in \mathrm{C}(X) \quad \frac{1}{N}\sum_{n=1}^{N} f(n^2\alpha) \overset{N\to\infty}{\longrightarrow} \int_0^1 f(t)\mathrm{d}t.$$

Hint: Let $\beta = 2\alpha$ and compute $T_\beta^n(\alpha, 0)$.

c. *(Weyl)* Let $p \in \mathbb{R}[X]$ a polynomial of degree at least 1 and such that $p' \notin \mathbb{Q}[X]$. Show that the sequence $\{p(n) + \mathbb{Z} \colon n \in \mathbb{N}\}$ equidistributes in the torus.

Hint: First, reduce to the case where $p = a_k X^k + \cdots + a_0$ with irrational leading coefficient a_k. Then proceed as follows. Let $\alpha = k! a_k$, look at the map $\Phi \colon \{\alpha\} \times \mathbb{T}^k \to \{\alpha\} \times \mathbb{T}^k$ given by

$$\Phi(\alpha, x_1, \ldots, x_k) = (\alpha, x_1 + \alpha, x_2 + x_1, \ldots, x_k + x_{k-1}),$$

and compute Φ^n.