Problem sheet 12

Problem 1

Let $\varphi\colon [0,1] \to [0,1]$ be given by

$$\varphi(x) = \begin{cases} 0 & \text{if } x = 0, \\ x \log x & \text{otherwise.} \end{cases}$$

a. Let $x, y, t \in [0,1]$. Show that

$$\varphi\big(tx + (1-t)y\big) \geq t\varphi(x) + (1-t)\varphi(y)$$

with equality if and only if $x = y$ or $t \in \{0,1\}$.

b. Let $k \geq 2$, $x_1, \dots, x_k \in [0,1]$, $t_1, \dots, t_k \in [0,1]$ such that $\sum_{i=1}^k t_i = 1$. Show that

$$\varphi\left(\sum_{i=1}^k t_i x_i\right) \geq \sum_{i=1}^k t_i \varphi(x_i)$$

with equality if and only if all x_i with $t_i > 0$ are equal, i.e.,

$$|\{x_i \colon 1 \leq i \leq k, t_i > 0\}| = 1.$$

c. Let $(X, \mathcal{B}, \mu)$ be a probability space and ξ a finite partition of $(X, \mathcal{B})$. Show that

$$-\sum_{A \in \xi} \varphi\big(\mu(A)\big) \leq \log|\xi|$$

with equality if and only if

$$\forall A \in \xi \quad \mu(A) = \frac{1}{|\xi|}.$$

Problem 2

Let $(X, \mathcal{B}, \mu)$ be a probability space and let ξ, η, and ζ be finite partitions of $(X, \mathcal{B})$. Show that the following are true.

a. $\xi \prec \eta \implies I_\mu(\xi|\eta) = 0$ μ-a.s.

b. $\xi \prec \eta \implies H_\mu(\xi|\eta) = 0$.

c. $I_\mu(\xi \vee \eta|\zeta) = I_\mu(\xi|\zeta) + I_\mu(\eta|\xi \vee \zeta)$ μ-a.s.

d. $H_\mu(\xi \vee \eta|\zeta) = H_\mu(\xi|\zeta) + H_\mu(\eta|\xi \vee \zeta)$.

e. $\xi \prec \eta \implies I_\mu(\xi|\zeta) \leq I_\mu(\eta|\zeta)$ μ-a.s.

f. $\xi \prec \eta \implies H_\mu(\xi|\zeta) \leq H_\mu(\eta|\zeta)$.

g. $\eta \prec \zeta \implies H_\mu(\xi|\eta) \geq H_\mu(\xi|\zeta)$.

h. $H_\mu(\xi \vee \eta|\zeta) \leq H_\mu(\xi|\zeta) + H_\mu(\eta|\zeta)$.

Problem 3

Let $(X, \mathcal{B}, \mu)$ be a probability space. Given $\mathcal{C}, \mathcal{D} \subseteq \mathcal{B}$ finite σ-algebras, we write $\mathcal{C} \mathring{\subseteq} \mathcal{D}$ if for every $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $\mu(C \Delta D) = 0$.

Given finite partitions $\xi, \eta \subseteq \mathcal{B}$ of $(X, \mathcal{B})$, we say that $\xi \mathring{=} \eta$ if $\sigma(\xi) \mathring{\subseteq} \sigma(\eta)$ and $\sigma(\eta) \mathring{\subseteq} \sigma(\xi)$.

Show that the following are true.

a. Suppose $\xi, \eta \subseteq \mathcal{B}$ are finite partitions of $(X, \mathcal{B})$. Then

$$\xi \mathring{=} \eta \implies H_\mu(\xi) = H_\mu(\eta).$$

b. Let $\mathscr{P}(\mathcal{B})$ denote the set of finite partitions of $(X, \mathcal{B})$. Show that $\mathring{=}$ defines an equivalence relation $\mathscr{P}(\mathcal{B})$.

c. Let $E = \mathscr{P}(\mathcal{B})/\mathring{=}$. Define

$$\begin{aligned} \varrho \colon E \times E &\longrightarrow [0, \infty), \\ \big([\xi], [\eta]\big) &\longmapsto \max\{H(\xi|\eta), H(\eta|\xi)\}. \end{aligned}$$

Show that ϱ is a metric on E.

Problem 4

Let $(X, \mathcal{B}, \mu)$ be a probability space and let ξ, η be finite partitions of $(X, \mathcal{B})$. One defines ξ and η to be *independent* if

$$\forall A \in \xi \forall B \in \eta \quad \mu(A \cap B) = \mu(A)\mu(B).$$

Show that ξ and η are independent if and only if $H_\mu(\xi|\eta) = H_\mu(\xi)$.

Problem 5

Let $T\colon \mathbb{T} \to \mathbb{T}$ be the $\times 2$-map, i.e., $T(x) = 2x$. Let $\xi = \{[0, 1/2), [1/2, 1)\}$.

a. Compute ξ_0^n for $n \in \mathbb{N}$.

b. Let μ be a Borel probability measure on $\mathbb{T}$. Show that $h_\mu(T, \xi) \leq \log 2$ with equality if and only if μ is the Lebesgue measure on $\mathbb{T}$.

Problem 6

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and ξ a partition of $(X, \mathcal{B})$. Show that

$$h_\mu(T, \xi) = \lim_{n\to\infty} H_\mu(\xi | \xi_0^n).$$