Problem sheet 2

Problem 1

Let X be a compact metric space and $T\colon X \to X$ continuous. The system (X,T) is called *(forward) topologically mixing* if for any pair $U,V \subseteq X$ of non-empty open sets there exists $n_0 \in \mathbb{N}$ such that

$$\forall n > n_0 \quad T^n(U) \cap V \neq \emptyset.$$

Prove that the doubling map

$$\begin{aligned} \times 2\colon \mathbb{T} &\longrightarrow \mathbb{T}, \\ x &\longmapsto 2x \bmod 1 \end{aligned}$$

is topologically mixing.

Problem 2

Let X_1 and X_2 be compact metric spaces and $T_i\colon X_i \to X_i$, $i = 1,2$, continuous. Consider the space $X = X_1 \times X_2$ with metric

$$d_X\big((x_1,x_2),(y_1,y_2)\big) = \max\big\{d_{X_1}(x_1,y_1), d_{X_2}(x_2,y_2)\big\}$$

and the map

$$\begin{aligned} T\colon X &\longrightarrow X, \\ (x_1,x_2) &\longmapsto \big(T_1(x_1), T_2(x_2)\big). \end{aligned}$$

Prove or disprove the following statements.

a. If T_1 and T_2 are topologically transitive, then T is topologically transitive.

b. If T_1 and T_2 are topologically mixing, then T is topologically mixing.

Problem 3

Find an example of a continuous map $T\colon \mathbb{T} \to \mathbb{T}$ such that

a. T is minimal but not topologically mixing.

b. T is topologically mixing but not minimal.

Problem 4

Let $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 - \{\mathbf{0}\}$. Show that

$$\mathcal{L}(\mathbf{v}) = \{t\mathbf{v} \bmod \mathbb{Z}^2 \colon t > 0\} \subseteq \mathbb{T}^2$$

is dense if and only if $v_2 \neq 0$ and v_1/v_2 is irrational.

Problem 5

Let X_1, X_2 be compact metric spaces and let $T_i \in \mathrm{Homeo}(X_)$, $i = 1, 2$. We call (X_1, T_1) and (X_2, T_2) semi-conjugate if there exists a surjective and continuous map $h\colon X_1 \to X_2$ such that $T_2 \circ h = h \circ T_2$.

Suppose that (X_1, T_1) and (X_2, T_2) are semi-conjugate. Show that the following statements are true.

a. If (X_1, T_1) is transitive, then so is (X_2, T_2).

b. If (X_1, T_1) is minimal, then so is (X_2, T_2).

c. For all $n \in \mathbb{Z}$ we have that $h\big(\mathrm{Fix}(T_1^n)\big) \subseteq \mathrm{Fix}(T_2^n)$.

Problem 6

Let X be a compact metric space and $T\colon X \to X$ continuous. For a subset $S \subseteq X$ we define the ω-limit of S as

$$\omega^+(S) = \big\{y \in X \colon \exists (y_k, n_k)_{k\in\mathbb{N}} \in (S \times \mathbb{N})^{\mathbb{N}}\, n_k \uparrow \infty \wedge y = \lim_{k\to\infty} T^{n_k}(y_k)\big\}.$$

a. Prove that $\omega^+(S)$ is T-invariant,

b. Prove that $\omega^+(S)$ is closed.

c. Find an example of a dynamical system, where

$$\overline{\bigcup_{x\in S} \omega^+(x)} \neq \omega^+(S).$$

Which inclusion does always hold?