Partial solutions to exercises for "Dynamical systems and Ergodic theory"

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1 Problem Sheet 1

Problem 1

Let

$$T\colon [0,1] \longrightarrow [0,1],$$
$$x \longmapsto \begin{cases} 0 & \text{if } x \in \{0,1\}, \\ nx-1 & \text{if there is } n \in \mathbb{N} \text{ so that } x \in [\frac{1}{n}, \frac{1}{n-1}). \end{cases}$$

1. Let $x \in [0,1]$. Show that
$$\exists \ell \in \mathbb{N} \quad T^\ell(x) = 0 \iff x \in [0,1] \cap \mathbb{Q}.$$
2. Show that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ is irrational.

Solution.

1. Suppose $x \in [0,1] - \mathbb{Q}$. Then $nx - 1 \notin \mathbb{Q}$ for every $n \in \mathbb{N}$. Hence $T(x) \notin \mathbb{Q}$. By induction, it follows that
$$\forall \ell \in \mathbb{N} \quad T^\ell(x) \notin \mathbb{Q}.$$
In particular, if $x \notin [0,1] \cap \mathbb{Q}$, then
$$\forall \ell \in \mathbb{N} \quad T^\ell(x) \neq 0.$$
Now suppose $x \in \mathbb{Q} \cap [0,1]$. Let $x = \frac{p}{q}$ with $p, q \in \mathbb{N}$ coprime. We can assume without loss of generality that $0 < x < 1$. Choose $n \in \mathbb{N} \backslash \{1\}$ such that $\frac{1}{n} \leqslant x < \frac{1}{n-1}$. Then
$$T(x) = \frac{np-q}{q}.$$
By choice of n we have that $np - q < p$, hence $T(x) < x$ and $T(x) = \frac{r}{s}$ with $r, s \in \mathbb{N}$ coprime such that $0 \leqslant r < s$ and $s \leqslant q$. Since the number of rationals in $[0,1]$ with denominator at most q is finite, $T^\ell(x) = 0$ after finitely many steps.
2. It suffices to show that
$$x = e - 2 = \sum_{k=2}^{\infty} \frac{1}{k!} \in [0,1]$$
is irrational. Note that
$$\forall n \in \mathbb{N} \quad \frac{1}{n} < \sum_{k=n}^{\infty} \frac{(n-1)!}{k!} = \frac{1}{n} + \frac{1}{n(n+1)} + \cdots < \sum_{k=1}^{\infty} \frac{1}{n^k} = \frac{1}{n} \frac{1}{1 - \frac{1}{n}} = \frac{1}{n-1}.$$
One computes that
$$T(x) = 2!x - 1 = \sum_{k=3}^{\infty} \frac{2}{k!}$$
and, hence, induction shows that
$$\forall n \in \mathbb{N} \quad T^n(x) = \sum_{k=n+2}^{\infty} \frac{(n+1)!}{k!} \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right).$$

Problem 2

Let $T_p \colon \mathbb{T} \to \mathbb{T}$ be the $\times p$-map.

1. Show that there exists $x \in \mathbb{T}$ with $\omega^+(x)$ uncountable but not $\mathbb{T}$, where
$$\omega^+(x) = \big\{y \in \mathbb{T} \colon \exists (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}} \text{ unbounded},\ y = \lim_{k\to\infty} T_p^{n_k}(x)\big\}.$$
2. Let $x_0 = \sum_{k=1}^{\infty} \frac{1}{p^{k!}}$ show that $\omega^+(x_0)$ is countable but not finite.

Solution.

1. Let $\Lambda = \bigcup_{n\in\mathbb{N}} \{0,2\}^n$ denote the set of all finite words in letters $\{0,2\}$. As argued in class, we can enumerate these words and then the symbols to construct a word whose forward-orbit approximates every word in the middle-third Cantor set arbitrarily well. The details are left to the reader.
2. We will provide an outline how one can show that $\omega^+(x_0) = \{p^{-m} \mod 1 \colon m \in \mathbb{N} \cup \{0\}\}$. Given $n \in \mathbb{N}$ let $k_n \in \mathbb{N}$ minimal such that $n \in \big[k_n!, (k_n+1)!\big)$. Then
$$\forall n \in \mathbb{N} \quad T_p^n(x_0) = p^n \sum_{k=k_n+1}^{\infty} \frac{1}{p^{k!}} \mod 1.$$
Note that for any $\ell \in \mathbb{N}$ we have
$$\sum_{k=\ell+1}^{\infty} \frac{1}{p^{k!}} < \frac{1}{p^{(\ell+1)!}} \sum_{k=0}^{\infty} \frac{1}{p^k} \leqslant \frac{2}{p^{(\ell+1)!}}.$$
In particular, it follows that for all sufficiently large $n \in \mathbb{N}$, we have that
$$T_p^n(x_0) \in \left[\frac{p^n}{p^{(k_n+1)!}}\left(1 - \frac{2}{p^{(k_n+1)!}}\right), \frac{p^n}{p^{(k_n+1)!}}\left(1 + \frac{2}{p^{(k_n+1)!}}\right)\right] \subset \mathbb{T}.$$
Now suppose that $x \in \omega^+(x_0)$ and let $(n_\ell)_{\ell\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be a w.l.o.g. strictly increasing sequence such that $x = \lim_{\ell\to\infty} T_p^{n_\ell}(x_0)$. Suppose first that for every $m \in \mathbb{N}$ the set
$$\Omega(m) = \big\{\ell \in \mathbb{N} \colon p^m = p^{(k_{n_\ell}+1)! - n_\ell}\big\}$$
is finite. This implies that
$$\lim_{\ell\to\infty} \frac{p^{n_\ell}}{p^{(k_{n_\ell}+1)!}} = 0.$$
It follows that for sufficiently large ℓ we have
$$0 \leqslant T_p^{n_\ell}(x_0) \leqslant< 2\frac{p^{n_\ell}}{p^{(k_{n_\ell}+1)!}} \xrightarrow{\ell\to\infty} 0.$$
Hence we can assume w.l.o.g. that there exists $m \in \mathbb{N}$ such that $\Omega(m)$ is infinite. Put differently, after passing to another subsequence, we can assume that
$$\forall \ell \in \mathbb{N} \quad T_p^{n_\ell}(x_0) \in \left[\frac{1}{p^m}\left(1 - \frac{2}{p^{(k_{n_\ell}+1)!}}\right), \frac{1}{p^m}\left(1 + \frac{2}{p^{(k_{n_\ell}+1)!}}\right)\right]$$
Since $k_{n_\ell} \to \infty$ as $\ell \to \infty$ this implies that $x = p^{-m}$.

Problem 3

Let A be a 2×2 real matrix with eigenvalues $\lambda \in (1,\infty)$ and $\mu \in (0,1)$. Consider the map
$$T \colon S^1 \longrightarrow S^1, \qquad x \longmapsto \frac{Ax}{\|Ax\|_2}.$$

1. Show that T has exactly four fixed points.
2. Show that $T^n x = \frac{A^n x}{\|A^n x\|_2}$.
3. Show that for every $x \in S^1$, the sequence $T^n x$ converges to a fixed point of T.

Solution.

1. Choose eigenvectors x_λ and x_μ in S^1 for eigenvalue λ and μ respectively. These exist since the eigenvalues are real and distinct. A quick computation shows that $T(x_\lambda) = x_\lambda$, i.e., x_λ is a fixed point and a similar argument shows the same for $-x_\lambda, x_\mu, -x_\mu$. So we have at least 4 fixed points. If x is a fixed point, then $T(x) = x$ is equivalent to $\|Ax\|_2 x = Ax$. Thus every fixed point of T is an eigenvector and there are only four eigenvectors that lie on the unit circle.
2. We prove the statement by induction. The case $n = 1$ is just the definition of our map. Now assume that $n \in \mathbb{N}\backslash\{1\}$ and suppose that the statement is true for $n-1$. Then
$$T^n(x) = T(T^{n-1}(x)) = T(\frac{A^{n-1}x}{\|A^{n-1}x\|_2}) = \frac{A\frac{A^{n-1}x}{\|A^{n-1}x\|_2}}{\|A\frac{A^{n-1}x}{\|A^{n-1}x\|_2}\|_2} = \frac{A^n x}{\|A^n x\|_2}.$$
3. Because our two eigenvectors are linearly independent, we can write every vector $x \in S^1$ as $x = \frac{\alpha x_\lambda + \beta x_\mu}{\|\alpha x_\lambda + \beta x_\mu\|_2}$ for some $\alpha, \beta \in \mathbb{R}$ uniquely determined up to positive scaling and not both equal to zero. If one of the factors α, β is zero, we have a fixed point. So we just need to look at the cases where both of the factors are non-zero. Then we can use part (b) and see that
$$\lim_{n\to\infty} T^n x = \lim_{n\to\infty} \frac{A^n x}{\|A^n x\|_2} = \lim_{n\to\infty} \frac{\lambda^n \alpha x_\lambda + \mu^n \beta x_\mu}{\|\lambda^n \alpha x_\lambda + \mu^n \beta x_\mu\|_2} = \lim_{n\to\infty} \frac{\alpha x_\lambda + (\frac{\mu}{\lambda})^n \beta x_\mu}{\|\alpha x_\lambda + (\frac{\mu}{\lambda})^n \beta x_\mu\|_2} = \operatorname{sgn}(\alpha) x_\lambda$$
The last equality holds because $(\frac{\mu}{\lambda}) < 1$ and hence $(\frac{\mu}{\lambda})^n$ converges to zero as $n \to \infty$.

Problem 4

1. Note that $d(x^{(\ell)}, x^{(1)}) \to 0$ as $\ell \to \infty$ if and only if $N(x^{(\ell)}, x^{(1)}) \to \infty$ as $\ell \to \infty$ which is the case if and only if x^ℓ and $x^{(1)}$ agree on a larger and larger number of initial symbols.
2. We apply Cantor's diagonal argument.
Let (x^ℓ) be a sequence in $X^{\mathbb{N}}$. Given $n \in \mathbb{N}$ and $1 \leqslant k < p$, let
$$\Omega_n(k) = \{\ell \in \mathbb{N}\colon x_n^{(\ell)} = k\}.$$
Since $\{0, \ldots, p-1\}$ is finite, there exists k_1 such that $\Omega_1(k)$ is infinite, i.e., there exists a subsequence $({}_1x^{(\ell)})_{\ell\in\mathbb{N}}$ of $(x^{(\ell)})$ such that ${}_1x_1^{(\ell)} = k_1$ for all $\ell \in \mathbb{N}$. Now we proceed by induction: Given $r \in \mathbb{N}$, $k_1, \ldots, k_r \in \{0, \ldots, p-1\}$, and a subsequence $({}_rx^{(\ell)})_{\ell\in\mathbb{N}}$ of $(x^{(\ell)})$ such that
$$\forall 1 \leqslant n \leqslant r \forall \ell \in \mathbb{N} \quad {}_rx_n^{(\ell)} = k_n,$$
choose $0 \leqslant k_{r+1} < p$ and a subsequence $({}_{r+1}x^{(\ell)})$ of $({}_rx^{(\ell)})$ such that
$$\forall \ell \in \mathbb{N} \quad {}_{r+1}x_{r+1}^{(\ell)} = k_{r+1}.$$
Define the subsequence $({}_*x^{(\ell)})$ of $(x^{(\ell)})$ by
$$\forall \ell \in \mathbb{N} \quad {}_*x^{(\ell)} = {}_\ell x^{(\ell)}.$$
Then ${}_*x^{(\ell)} \to (k_n)_{n\in\mathbb{N}}$ as $\ell \to \infty$.
3. A point $x \in X$ is a fixed point for σ^n if and only if there exists $n \in \mathbb{N}$ such that
$$\forall k \in \mathbb{N} \quad x_k = x_{k+n}.$$
Then x is certainly periodic with period at most n and x has period exactly n if and only if
$$x \in \operatorname{Fix}(\sigma^n)\backslash \bigcup_{1\leqslant\ell<n} \operatorname{Fix}(\sigma^\ell). \tag{1}$$
We need to make this more tractable.
First, we note that for $1 \leqslant \ell \leqslant n$, if a point $x \in \operatorname{Fix}(\sigma^n)$ is ℓ periodic (with exact period ℓ), then $\ell | n$. To this end, we will show that x is eventually (ℓ, n)-periodic.
Let $a, b \in \mathbb{Z}$ such that $an + b\ell = (n, \ell)$. For any $k > |an| + |b\ell|$ we have that $k + an \geqslant |bn| + 1$ and, hence,
$$x_k = x_{k+an} = x_{k+an+b\ell} = x_{k+(n,\ell)}.$$

Hence x is eventually (n, ℓ)-periodic. Since x is periodic, it is (n, ℓ)-periodic. It follows that

$$\mathrm{Per}(\sigma, n) = \mathrm{Fix}(\sigma^n) \setminus \bigcup_{\substack{\ell \mid n \\ \ell < n}} \mathrm{Per}(\sigma, \ell).$$

Hence, if we denote $g(n) = |\mathrm{Per}(\sigma, n)|$, we get the recursive formula

$$g(n) = p^n - \sum_{\substack{\ell < n \\ \ell \mid n}} g(\ell).$$

4. Let $x \in X$ and $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\forall y \in X \quad N(x, y) > N \implies d(x, y) < \varepsilon.$$

Define x^* by

$$\forall 1 \leqslant n \leqslant N \quad x^* = x_n$$

and continue periodically by requiring $x \in \mathrm{Fix}(\sigma^N)$.