Exercise 1.1. ♣
Which of the following statements are true? There may be more than one true statement.

(a) If $f \in L^1(\mathbb{R}^n)$ then necessarily $f \in L^2(\mathbb{R}^n)$?

(b) If $f \in \ell^1(\mathbb{N})$ then necessarily $f \in \ell^2(\mathbb{N})$?

(c) Let $(f_n)_{n\in\mathbb{N}}$ be a collection of non-negative measurable functions. Is it true that

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n(x)\,dx\,?$$

(d) Is it possible to find a sequence of functions $\{f_k\} \subset L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |f_k(x) - 1|^2\,dx \to 0 \text{ as } k \to \infty\,?$$

(e) A Cauchy sequence (say, in a metric space) can have at most one limit point?

(f) The interval $(0,1) \subset \mathbb{R}$ is complete?

Exercise 1.2.
Consider the sequence of functions $f_n\colon (0,\infty) \to \mathbb{R}$ defined by

$$f_n(x) = \frac{\arctan(x)}{x}\chi_{(0,n)}(x).$$

Determine the pointwise limit f and discuss the convergence $f_n \to f$ in $L^2\big((0,\infty)\big)$. Is the limit also in $L^1\big((0,\infty)\big)$? What can we deduce about the completeness of the space L^1 with respect to the norm $\|\cdot\|_{L^2}$?

Exercise 1.3.
Recall that the Dominated Convergence Theorem implies that a collection of measurable functions $f_n\colon \mathbb{R} \to \mathbb{C}$, satisfying $|f_n| \le g$ for some $g \in L^1(\mathbb{R})$, also satisfies

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} \Big(\lim_{n\to\infty} f_n(x)\Big)\,dx$$

whenever the pointwise limit $\lim_{n\to\infty} f_n(x)$ exists a.e. Show, via a counterexample, that the hypothesis $|f_n| \le g \in L^1(\mathbb{R})$ is necessary.

Hint: Can you think of an example in which the statement fails? For instance, a sequence of functions f_n with constant integral (> 0) but with poitwise limit 0?

Exercise 1.4.
Consider the space $C\big([0,1]\big)$ of continuous functions on $[0,1]$ equipped with the norm

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|.$$

Show that this is not an inner product space, i.e. there is no inner product on $C\big([0,1]\big)$ which induces this norm.

Hint: Recall the parallelogram law.

Exercise 1.5.
Consider the space $\ell^p(\mathbb{N})$ of p-summable sequences (for $p \geq 1$) with the norm

$$\|x\|_p = \Big(\sum_{n\in\mathbb{N}} |x_n|^p\Big)^{\frac{1}{p}}.$$

Show that $\ell^p(\mathbb{N})$ with $p \neq 2$ is not an inner product space.