Exercise 2.1.
(a) Let $V := \mathrm{M}_{n\times n}(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries and define the *Fobenius product* $\langle\cdot,\cdot\rangle\colon V \times V \to \mathbb{C}$ as

$$\langle A, B\rangle := \mathrm{Tr}(AB^\dagger) = \sum_{i,j=1}^{n} a_{ij}\bar{b}_{ij}$$

where Tr denotes the trace and $B^\dagger$ is the Hermitian transpose of B, obtained by transposition and complex conjugation of the entries: $B^\dagger = \overline{B^T}$. Show that $(V, \langle\cdot,\cdot\rangle)$ is an inner-product space.
Hint: first observe that $\mathrm{Tr}(A) = \overline{\mathrm{Tr}(A^\dagger)}$.

(b) Consider n inner-product spaces $(V_1, \langle\cdot,\cdot\rangle_1), \dots, (V_n, \langle\cdot,\cdot\rangle_n)$. Show that $(V, \langle\cdot,\cdot\rangle)$, where $V = V_1 \times \cdots \times V_n$ and

$$\langle(v_1, \dots, v_n), (w_1, \dots, w_n)\rangle := \sum_{i=1}^{n} \langle v_i, w_i\rangle_i,$$

is an inner product space.

(c) Let $W := M_{n\times n}(L^2(\mathbb{R}, \mathbb{C}))$ be the space of $n \times n$ matrices whose entries are square integrable functions from $\mathbb{R}$ to $\mathbb{C}$. Which product would make W an inner product space?
Hint: observe that W is a "composition" of two inner product spaces.

Exercise 2.2.
An inner product space $(V, \langle\cdot,\cdot\rangle)$ is also a metric space under the norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$, hence it has a natural topology (induced by the metric $d(v, w) = \|v - w\|$). Prove that the vector space operations $+ : V \times V \to V$ and $\cdot : \mathbb{C} \times V \to V$ and the inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ are continuous, where $V \times V$ and $\mathbb{C} \times V$ are endowed with the natural product topologies. Recall that the topology in $V \times V$ is the one induced by the norm

$$\|(v_1, v_2)\|_{V\times V} := \|v_1\| + \|v_2\|.$$

Similarly, the norm (thus the metric and the topology) on $\mathbb{C} \times V$ is given by

$$\|(\alpha, v)\|_{\mathbb{C}\times V} := |\alpha| + \|v\|.$$

Exercise 2.3.
Show that $\ell^p(\mathbb{N})$ for $0 < p < 1$ is a vector space but not a normed space, that is $\left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{\frac{1}{p}}$ does not define a norm on $\ell^p(\mathbb{N})$.

Exercise 2.4. ♣
In the normed space $(L^2((0,1)), \|\cdot\|_{L^2})$, consider the subset

$$X = \left\{ f \in L^2((0,1)) : \int_0^1 f \, dx = 1 \right\}.$$

Which of the following statements are true?

- ☐ X is not well-defined.
- ☐ X is well-defined, open and convex.
- ☐ X is well-defined, closed, convex but not a linear subspace.
- ☐ X is well-defined, closed and a linear subspace.

Exercise 2.5.
In the following normed spaces, determine whether the given subsets X are well-defined, open, closed, linear subspaces and/or convex.
(a) In the normed space $(C([0,1]), \|\cdot\|_{L^\infty})$, the subset X of nowhere vanishing functions.

(b) In the normed space $(C([0,1]), \|\cdot\|_{L^2})$, the subset X of nowhere vanishing functions.

(c) In the normed space $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$, the subset

$$X = \left\{ f \in L^2(\mathbb{R}) : f(x) = f(-x) \text{ for a.e. } x \in \mathbb{R} \right\}.$$

Hint: It's useful to recall that if $u_k \to u$ in L^2 then, up to picking a subsequence, there is a null measure set N such that $u_k(x) \to u(x)$ for all $x \notin N$.

(d) (★) In the normed space $(L^2((0,1)), \|\cdot\|_{L^2})$, the subset

$$X = \left\{ f \in L^2((0,1)) : f \geq 0 \text{ a.e. and } \int_0^1 \frac{2f}{1+f} \, dx \geq 1 \right\}.$$

Hint: observe that the map $s \mapsto 2s/(1+s)$ is concave for $s \geq 0$.