Exercise 3.1. ♣
Which of the following pairs (vector space, bilinear form) are Hilbert spaces?
(a) $V := L^2(\mathbb{R};\mathbb{C})$ and $\langle u,v\rangle := \int_{\mathbb{R}} u(t)\bar{v}(t)\,\frac{dt}{1+t^2}$

(b) $V := \{$real polynomials of degree at most $N\}$ and $\langle p,q\rangle := p(\frac{d}{dx})|_{x=0}\, q$.

Hint: If $p(X)$ is a polynomial, then $p(\frac{d}{dx})|_{x=0}$ is the differential operator obtained by replacing X with $\frac{d}{dx}$ and then evaluating at $x=0$. Example: if $p(X) = X^2+3$ then $p(\frac{d}{dx})|_{x=0}q = q''(0)+3q(0)$. Observe that $\left(\frac{d}{dx}\right)^j_{x=0} x^k = \delta^{kj}\,k!$.

(c) $V := L^1\big((0,1);\mathbb{R}\big)$ and $\langle u,v\rangle := \int_0^1 u(x)v(x)\,dx$.

(d) $V := \mathbb{Q}^d$ and $\langle x,y\rangle := \sum_{k=1}^d x_k y_k$.

Exercise 3.2.
Let

$$V := \big\{u \in C^2((0,1)) \cap C([0,1]) : u', u'' \text{ bounded on } (0,1),\ u(0)=0\big\}$$

Prove or disprove that the following maps $\|\cdot\|\colon V \to \mathbb{R}$ are norms (no need to check completeness) and determine whether they arise from an inner product.
(a) $\|u\| = \left(\int_0^1 |u''(x)|^2 dx\right)^{1/2}$

(b) $\|u\| = \left(\int_0^1 |u'(x)|^2 dx\right)^{1/2}$

(c) $\|u\| = \left(\int_0^1 |u'(x)|^3 dx\right)^{1/3}$

(d) $\|u\| = \left(\int_0^1\int_0^1 \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy\right)^{1/2}$
Hint: Recall the Minkowski inequality: for $p\in(1,+\infty)$ and $f,g\in L^p(X,\mu)$, we have $(\int_X |f+g|^p d\mu)^{1/p} \le (\int_X |f|^p d\mu)^{1/p} + (\int_X |g|^p d\mu)^{1/p}$.

Exercise 3.3.
Consider the Hilbert space $H := L^2\big((-1,1)\big)$. Apply the Gram-Schmidt algorithm to the ordered set $\{1,x,x^2\}\subset H$, and find three orthonormal polynomials $e_0(x), e_1(x), e_2(x)$.

Exercise 3.4.
Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space. Show that a linear subspace of H is itself a Hilbert space with respect to $\langle\cdot,\cdot\rangle$ if and only if it is closed.

Exercise 3.5. ★

This exercise is concerned with a quantitative study of the Cauchy-Schwarz inequality.

(a) Let H be a real inner product space. We write $x \cdot y$ for the inner product of $x, y \in H$ and $|x|$ for the induced norm. Prove the following identity:

$$|x||y| - x \cdot y = \frac{|x||y|}{2}\Big|\frac{x}{|x|} - \frac{y}{|y|}\Big|^2 \geq 0, \quad \forall\, x, y, \in H.$$

(b) Characterize the set $C \subset H \times H$ of pairs of vectors that saturate the Cauchy-Schwarz inequality, i.e. $x \cdot y = |x||y|$. Plot C in the case $H = \mathbb{R}$.

(c) If x, y are ϵ-close to saturating the Cauchy-Schwarz inequality, that is

$$x \cdot y \geq (1 - \epsilon)|x||y|,$$

then how close are x, y to the set C? Find an upper bound for the quantity

$$\operatorname{dist}\big((x, y), C\big)^2 := \inf_{(x', y') \in C} |x - x'|^2 + |y - y'|^2.$$