Exercise 4.1. ♣
Which of the following statements are true?
(a) $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ are isometrically isomorphic as Hilbert spaces.

(b) The projection of the element $x = \left(\frac{n}{(n+1)^2}\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ onto the subspace generated by $y = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ is given by $(\frac{\pi^2}{6} - 1)y$
Hint: You can use $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(c) The projection of the element $x = \left(\frac{n}{(n+1)^2}\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ onto the subspace generated by $y = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ is given by $(1 - \frac{6}{\pi^2})y$
Hint: You can use $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(d) Given any $u \in L^2((0,1))$ there exists a unique polynomial $\tilde{p}$ that minimizes the following function on the set of polynomials: $p \mapsto \|u - p\|_{L^2((0,1))}$.
Hint: Recall that the set of polynomials is dense in $L^2((0,1))$.

(e) Let H be a Hilbert space. If $K \subset H$ is not convex, then there might not be a unique projection $\pi_K(x)$ onto K.

Exercise 4.2.
Consider the Hilbert space $H = L^2((-1,1))$. For each of the following subspaces $Y \subset H$ show that Y is closed, find its orthogonal complement $Y^\perp$ and find a formula for the projection $\pi_Y\colon H \to Y$.
(a) $Y = \{u \in H : u = \text{constant a.e.}\}$.

(b) $Y = \{u \in H : \int_{-1}^{1} u(x)\, dx = 0\}$.

(c) $Y = \{u \in H : u(x) = u(-x) \text{ a.e.}\}$.

Exercise 4.3.
Calculuate the minimum

$$\min_{a,b,c\in\mathbb{C}} \int_{-1}^{1} \left|x^3 - ax^2 - bx - c\right|^2 dx.$$

Hint: Recall Exercise 3.3 on Problem Sheet 3.

Exercise 4.4.
(a) Let H be a Hilbert space and $K \subset H$ a closed convex subset. Show that the projection onto K satisfies

$$\|\pi_K(x) - \pi_K(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Hint: Consider the degree 2 polynomial $p(t) = \|(1-t)\pi_K(x) + tx - (1-t)\pi_K(y) - ty\|^2$ and its derivative at $t = 0$.

(b) Let now $(V, \|\cdot\|)$ be a normed vector space and define the map

$$F : V \to V, \quad F(x) = \begin{cases} x & \text{if } \|x\| \leq 1 \\ \frac{x}{\|x\|} & \text{if } \|x\| > 1 \end{cases}$$

Show that

$$\|F(x) - F(y)\| \leq 2\|x - y\|, \quad \forall x, y \in V.$$

(c) Show that, in general, the constant 2 in the bound above cannot be improved.
Hint: Take $V = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$.

(d) What happens if V is a Hilbert space and $\|\cdot\|$ its Hilbert norm? Can the bound be improved?

Exercise 4.5.
(a) Show that for $1 \leq p < \infty$ the space $\ell^p_{\mathbb{R}}(\mathbb{N})$ is separable.
Hint: Consider the set

$$S = \{x = (x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}) : x_n \in \mathbb{Q}\ \forall\, n \in \mathbb{N},\ x_n = 0 \text{ except for finitely many } n\}.$$

(b) The space $\ell^\infty_{\mathbb{R}}(\mathbb{N})$ is not separable.
Hint: Consider the set

$$S = \{x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}) : x_n \in \{0, 1\}\ \forall n \in \mathbb{N}\}$$