Exercise 5.1. ♣
Which of the following statements are true?
(a) If a linear map $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfies $\|Tu\|_{L^2(\mathbb{R})} \leq 100$ for all $u \in L^2(\mathbb{R})$ with $\|u\|_{L^2(\mathbb{R})} \leq \frac{1}{10}$, then T is continuous.

(b) Assume that two bounded linear functionals $\phi, \psi \in L^2([0,1])^*$ agree on $C([0,1])$, i.e. $\phi(u) = \psi(u)$ for all $u \in C([0,1]) \subset L^2([0,1])$. Then $\phi = \psi$.

(c) If ϕ is a continuous linear functional on a Hilbert space H, then $\ker \phi$ is a closed linear subspace of H.

(d) For $x = (x_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$ define $(Tx)_k = \log(1/k)x_k$ for all $k \in \mathbb{N}$. Then T defines a bounded linear operator $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$.

(e) For $u \in L^2((0,1))$ define $F(u)(x) = u(x)^2$. Then $F : L^2((0,1)) \to L^2((0,1))$ defines a bounded (non-linear) operator.

Exercise 5.2.
For a fixed measurable function $a\colon (0,1) \to \mathbb{C}$, consider the multiplication operator

$$M_a : L^2((0,1)) \to L^2((0,1)), \quad M_a u(x) = a(x)u(x).$$

We want to prove that M_a is continuous on $L^2((0,1))$ if and only if $a \in L^\infty(0,1)$, in which case the operator norm satisfies $\|M_a\|_{\mathrm{op}} = \|a\|_{L^\infty(0,1)}$.
(a) Prove the inequality

$$\int_0^1 |a(x)u(x)|^2\,dx \leq \operatorname{esssup}_{x\in(0,1)}|a(x)|^2 \int_0^1 |u(x)|^2\,dx,$$

and deduce that $\|M_a\|_{\mathrm{op}} \leq \|a\|_{L^\infty(0,1)}$.

(b) Show that if $E \subset (0,1)$ is measurable with $|E| > 0$, then

$$\frac{\|M_a \mathbf{1}_E\|^2_{L^2(0,1)}}{\|\mathbf{1}_E\|^2_{L^2(0,1)}} = \frac{1}{|E|}\int_E |a(x)|^2\,dx.$$

(c) By an appropriate choice of the measurable set E in the previous point, prove that $\|M_a\|_{\mathrm{op}} \geq \|a\|_{L^\infty}$.

Hint: Take $E =$“the set where $|a|$ is large” and recall the definition of essential supremum.

Exercise 5.3.
Prove that each of the following linear operators is bounded on $\ell^2(\mathbb{N})$ (i.e. as an operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$). Illustrate each operator as an infinite matrix with respect to the standard basis vectors $e_n = (\delta_{n,k})_{k\in\mathbb{N}}$.
(a) (Shift operator) $S\colon (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$.

(b) (Diagonal matrix) $M_\lambda\colon (x_1, x_2, x_3, \ldots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \ldots)$, where $\{\lambda_k\}_{k\geq 1}$ is some given sequence such that $\sup_{k\geq 1} |\lambda_k| < \infty$.

(c) $T\colon (x_1, x_2, x_3, \ldots) \mapsto (x_1 - x_2, x_2 - x_3, x_3 - x_4, \ldots)$.

(d) (Hilbert-Schmidt matrix) For each $k \geq 0$ set $(Ax)_k := \sum_{j\geq 1} A_{k,j} x_j$, where the infinite matrix $\{A_{i,j}\}_{i\geq 1, j\geq 1}$ satisfies

$$\sum_{i,j\geq 1} |A_{i,j}|^2 < \infty.$$

Hint: Apply the Cauchy-Schwarz inequality to the sum $\left|\sum_{j\geq 1} A_{k,j} x_j\right|^2$ for fixed k.

Exercise 5.4.
Prove the following inequalities and interpret them as the continuity of a suitable linear map between suitable normed vector spaces:
(a) For all $u \in L^2(\mathbb{R})$, we have

$$\int_0^1 |u(t)|^2 \, dt \leq \int_{\mathbb{R}} |u(t)|^2 \, dt.$$

(b) For each polynomial $p(X) = p_0 + p_1 X + \ldots + P_k X^N$, we have

$$\max_{x\in[-1,1]} |p(x)| \leq \sum_{j=0}^{N} |p_j|.$$

(c) For all $u \in C^1([0,1])$ with $u(0) = 0$, we have

$$\max_{x\in[0,1]} |u(x)| \leq \int_0^1 |u'(t)| \, dt.$$

Exercise 5.5.

This exercise concerns compactness in infinite dimensional Hilbert spaces.

(a) Show that the closed unit ball $\overline{B_1^{\ell^2(\mathbb{N})}} = \left\{x \in \ell^2(\mathbb{N}) : \|x\|_{\ell^2} \leq 1\right\}$ is not a compact subset of $\ell^2(\mathbb{N})$.

Hint: Find a sequence contained in the closed unit ball with no converging subsequences.

(b) Show that $C = \left\{x = (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : |x_n| \leq \frac{1}{n} \ \forall n \in \mathbb{N}\right\}$ is a compact subset of $\ell^2(\mathbb{N})$. C is known as the Hilbert cube.

Hint: Show that C is totally bounded, i.e. C can be covered by finitely many balls of any fixed radius.