Exercise 6.1. ♣
For each of the following Hilbert spaces H and maps $\varphi : H \to \mathbb{C}$ determine whether φ defines a continuous linear functional on H.
(a) $H = L^2([-\pi, \pi])$ and $\varphi(f) = c_1(f)$ (the first Fourier coefficient of f).

(b) $H = L^2([-1, 1])$ and $\varphi(f) = f(0)$.

(c) $H = \ell^2(\mathbb{N})$ and $\varphi((x_n)_{n\in\mathbb{N}}) = x_3 + 2x_7$.

(d) $H = L^2([-1, 1])$ and $\varphi(f) = \int_{-1}^{1}(1 + f)^2\, dx$.

(e) $H = L^2(\mathbb{R})$ and $\varphi(f) = \frac{1}{3}\int_{-1}^{1} f\, dx$.

(f) $H = \ell^2(\mathbb{N})$ and $\varphi((x_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}$.

Exercise 6.2.
Show that any $f \in C^1([-\pi, \pi])$ with $f(\pi) = f(-\pi)$ and $\int_{-\pi}^{\pi} f(x)\, dx = 0$, satisfies

$$\|f\|_{L^2([-\pi,\pi])} \le \|f'\|_{L^2([-\pi,\pi])}.$$

Exercise 6.3.
Let $f \in C(\mathbb{R})$ be a 2π-periodic continuous function satisfying $c_0(f) = 0$.
(a) Show that $F(t) = \int_0^t f(x)\, dx$ is also a 2π-periodic function and determine its Fourier coefficients $c_n(F)$ for all $n \neq 0$.

(b) Determine the Fourier coefficient $c_0(F)$ in terms of the $c_n(f)$.

Exercise 6.4.
(a) Is there an element of $L^2((0, 2\pi))$ whose Fourier series is

$$\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log(n)}\,?$$

(b) Show that the sequence above converges pointwise for each $x \in (0, 2\pi)$.

Hint: Use Dirichlet's test.

Exercise 6.5.

The purpose of this exercise is to prove the Riemann–Lebesgue lemma in the special case of a characteristic function of a bounded interval in $\mathbb{R}$.

Let $p : \mathbb{R} \to \mathbb{C}$ be a bounded T-periodic function. Let $f = \mathbf{1}_{[a,b]}$ be the characteristic function of a bounded interval $[a,b] \subset \mathbb{R}$ for some $a < b$. Show that

$$\lim_{x \to \pm\infty} \int_{\mathbb{R}} f(t)p(xt)\,dt = \mu \int_{\mathbb{R}} f(t)\,dt, \tag{1}$$

where μ is the average of p over one period

$$\mu = \frac{1}{T} \int_0^T p(t)\,dt.$$

Remark: The Riemann–Lebesgue lemma says that (1) in fact holds for any $f \in L^1(\mathbb{R})$. How might one use this exercise to prove the more general version?

Exercise 6.6.

Use the Riemann-Lebesgue lemma (see the Remark in Exercise 6.5 or Lemma 2.32 in the lecture notes) to compute the following limits.

(a) Let $A \subset \mathbb{R}$ be a Lebesgue measurable set with finite Lebesgue measure $|A| < \infty$. Compute

$$\lim_{m \to \infty} \int_A \sin^2(mx)\,dx.$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic of period 1. Compute

$$\lim_{\epsilon \to 0^+} \sqrt{\epsilon} \int_{\mathbb{R}} f(x)e^{-\epsilon\pi x^2}\,dx.$$